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ABSTRACT

The Sequential Mathematics Sequence provides an integrated course of study for grades 9-11 in New York City. This document presents lesson plans for part one of the first-year course, designed for ninth grade. The curriculum interweaves algebra, geometry, logic, probability, statistics, and trigonometry, with the emphasis in the 60 lessons in this document on logic, algebra, probability, and statistics. Each lesson begins with a clearly stated aim presented in question form. Performance objectives, new vocabulary, and a motivational activity follow. A full development is then presented, with suggested pivotal questions. Applications, a summary, and sample homework examples are also included. In each lesson opportunities are infused for helping students to develop critical thinking, problem-solving, and reasoning skills. (MNS)

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INTEGRATED MATHEMATICS

Course I

Term I

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INTEGRATED MATHEMATICS

Course I

Term I

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NEW YORK CITY BOARD OF EDUCATION
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Division of Curriculum and Instruction

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FOREWORD

The Integrated Mathematics Sequence provides New York City students and teachers with an integrated course of study for grades nine, ten, and eleven. In a departure from traditional courses of study, lessons interweave algebra, geometry, logic, probability, and statistics. The Integrated Mathematics Sequence extends the integrated approach that begins in kindergarten and continues through grade 8.

In the Integrated Mathematics Sequence, algebra, geometry, and trigonometry remain important components of the curriculum. In addition, other branches of mathematics -- such as probability and statistics -- are introduced and emphasized as important tools for dealing with present and future technological developments.

Each lesson in Course I begins with a clearly stated aim presented in question form. Performance objectives follow; these provide a specific focus for the teacher. The new vocabulary for the lesson is presented. This is followed by a motivational activity. A full development is presented, with suggested pivotal questions. Applications, a summary, and sample homework examples are also included. In each lesson opportunities have been infused for helping students to develop critical thinking, problem-solving, and reasoning skills. These activities will enable students to become more skilled in higher levels of thinking.

The Integrated Mathematics Sequence has been designed to anticipate students' needs for mathematical skills in an increasingly technologically-oriented culture. We anticipate that the Integrated Mathematics Sequence will serve the individual needs of your students and enable them to acquire the mathematics skills they need in the present and for the future.

Charlotte Frank

Charlotte Frank
Executive Director
Division of Curriculum and Instruction

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Morris Freedman, Director of the Office of Curriculum Development and support, provided overall supervision of this project.

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INTRODUCTION

Integrated Mathematics (Course I) is the first in a proposed series of publications offering teachers of mathematics an orderly set of topics and objectives for grades nine, ten, and eleven. This course is consistent with the proposed New York State Curriculum in Integrated Mathematics.

Traditionally, in New York State, mathematics in grades nine, ten, and eleven has consisted of algebra, geometry, and intermediate algebra/trigonometry. Students completing the three-year sequence have been exposed to these topics as relatively independent units. Moreover, students discontinuing their study of mathematics after grade eleven have never been exposed to such timely and practical topics as probability, statistics, and formal logical reasoning. Those terminating their high school mathematics program after grade ten have been given even fewer opportunities for exposure to these topics.

This curriculum guide is based upon the Scope and Sequence for Integrated Mathematics, Course I, Part 1. It consists of lesson plans, which should be considered suggestions rather than prescriptions. Each lesson stresses the development of mathematical concepts, with an emphasis on providing opportunities for the development of problem solving skills, reasoning skills and critical thinking. To help students develop these skills, each lesson contains suggested questions to encourage student investigation and thought. The problems presented in the lessons are designed to be relevant to students' experiences. There is an abundance of application and homework problems. These have been provided to help teachers adapt instruction to the individual differences within the classroom.

Though Course I is written with ninth grade students in mind, it is possible that some students may begin the material earlier or later, or complete it in less than or more than one year. It is expected that individual teachers will consider their students' abilities and interests in making judgments concerning time allotments, aspects of material to be stressed, or extra material to be presented.

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Lesson 1 - Logic and Statements

AIM: What is logic? What is a statement?

PERFORMANCE OBJECTIVES: The students will be able to...

1. Describe what is meant by logic.
2. Define each of the following terms:
    a) sentence
    b) open sentence
    c) variable
    d) statement
    e) domain
    f) solution set
3. Distinguish between statements and non-statements.
4. Distinguish between an open sentence and a statement.
5. Determine the truth or falsity of many statements.
6. Find the solution set for open sentences given the domain of their variables.

VOCABULARY: logic, sentence, statement, open sentence, variable, domain, solution set

MOTIVATION: What similarities and differences do you observe in these sentences:

1. Central Park is the largest park in Manhattan.
2. Abraham Lincoln was the second president of the U.S.
3. He is wearing a blue shirt.

Students should observe:
the first two are about specific things (Central Park, Abraham Lincoln);
the third is about an unidentified boy or man;
the first is definitely true;

the second is definitely false;
the third may be true or false, depending on which boy or man is considered.

DEVELOPMENT (Include pivotal questions):

1. Elicit various definitions of logic. Write the definition: Logic is the study of reasoning. Logic helps us determine the truth or falsity of a sentence.
2. Give the following sentences and ask whether they are true or false.
   a) Every square has four sides.
   b) The day after Monday is Tuesday.
   c) The longest month of the year is June.
   d) $6 + 7 = 13$
   e) $8 \div 2 = 3$

   Conclude that all of these are sentences whose truth can be determined. These are called statements.
3. Now present:
   a) x is greater than 5.
   b) He has brown hair.
   c) It is made of metal.

   Ask pivotal question: What is the difference between these sentences and those above?
   Elicit: The truth of these sentences cannot be determined without additional information.
4. Introduce and define these as open sentences.
   Definition: An open sentence is a sentence whose truth cannot be determined since it contains an unknown quantity. This is called a variable (e.g., x, he, it).
5. Refer to 3a.
   Pivotal question: What are possible answers?
   Discuss the fact that these are determined by the choices which are available. Hence the concept of domain is introduced.

Definition: The domain of a variable is the set of values that may be used for the variable.

For $x > 5$, if $D = \{\text{integers}\}$, the answer is $\{6,7,8...\}$. If the domain $= \{\text{even integers}\}$, the answer is $\{6,8,10...\}$, and so on. These sets are each called the solution sets.

Definition: A solution set is the set of all permissible values that make an open sentence true.

Pivotal question: Why can the solution set for an open sentence vary?

APPLICATIONS:

A) DIRECTIONS

1. State whether each sentence is an open sentence or a statement.
2. For each statement, determine whether it is true or false.
3. For each open sentence, state the variable and replace it to make a true statement.

a) $7 + 3 = 5 + 5$

b) The day after Thursday is Wednesday.

c) $y + 6 = 18$

d) It has three sides.

e) He is six feet tall.

f) The Bronx is in New York.

g) It is a city in Connecticut.

h) 3 times 5 is less than 12.

i) x is an odd number.

B) 1. Given the domain for each variable is the set of whole numbers $\{0,1,2,3,...\}$, find the solution set for each of the following.

a) $x + 7 = 12$

b) $y - 2 = 6$

c) Two times a number is greater than 5.

d) x is less than 4.

e) z is an odd number.

2. For each of the above, write the solution set if the domain is $\{2,4,6,8,10\}$. (The teacher should note the possibility of the empty set as a solution set.)

C) 1. If the domain for each variable is {Monday, Tuesday, Wednesday, March, April, June, September}, find the solution set for each of the following sentences.
   - a) It begins with the letter M.
   - b) It is a day that comes after Monday.
   - c) It contains 30 days.
   - d) The word contains 6 or fewer letters.
   - e) It is a day of the week.

SUMMARY:

1. What is logic?
2. Why can't we determine the truth of all sentences?
3. How does the domain of the variable affect the solution set of an open sentence?
4. Give an example of a statement which is true.
5. Give an example of a statement which is false.
6. Give an example of an open sentence which is always true (true for every possible replacement of the variable).
7. Give an example of an open sentence which is never true.

HOMEWORK:

1. State whether each of the following is an open sentence, a statement, or neither.
   - a) The formula for the area of a circle is $A = \pi R^2$.
   - b) $\pi = \frac{22}{7}$
   - c) The Greatest Show on Earth.
   - d) Who is president of the United States?
   - e) Zero is neither even nor odd.
   - f) $x + x = 2x$ whenever x is a number.
   - g) $x + y = 6$
   - h) $x + y = y + x$

2. If the domain of each variable is the set of whole numbers, find the solution set for each of the following sentences.

a) $x^2 < 10$

b) $2a + 3 = 11$

c) p is a prime number between 10 and 20.

d) $x^2 + y^2 = 0$

e) $(n - 5) + 4 = 1$

f) $r + s = 4$

g) $\frac{y}{y} = 1$

*h) $\frac{1}{a} + \frac{1}{b} = \frac{1}{2}$

*i) $a - 3$ is not defined.

*j) $a^2 + b^2 = c^2$ and $c < 10$.

Lesson 2 - Negation and Truth Tables

AIM: 1) What is the negation of a statement?
2) How do we construct a truth table?

PERFORMANCE OBJECTIVES: The students will be able to...

1. Use a symbol to represent a statement.
2. Form the negation of a statement in both words and symbols.
3. Construct the truth table for negation.
4. Given an open sentence in symbolic notation using P, determine when the statement that results from substituting given statements for the statement variable is true, false or open.
5. Given an open sentence and the domain of its variable, negate it and find its solution set.

VOCABULARY: negation, truth table

MOTIVATION:

Is the statement "Four times two = 7" true? How could we make this a true statement? (Students may suggest $3\frac{1}{2}$ times two = 7, four plus three = 7, etc.) Then ask: Could we make this a true statement without changing any of the words, symbols or numerals of the original statement?

DEVELOPMENT (Include pivotal questions):

1. Elicit that the insertion of a negative such as not would make this a true statement. For example, the sentence "4 times 2 is not = 7" is true.
   Ask pivotal question: What is the relationship between the original sentence and the new sentence?

Elicit: One is the negative of the other.
These statements are called negations of each other.

2. Present these sentences and have the students form the negation of each. Have them determine the truth or falsity of each sentence and of its negation.
   a) The day after Monday is Tuesday.
   b) A square has three sides.
   c) 4 + 7 does not equal 11.
   d) Elephants like to eat peanuts.
   e) Michael Jackson did not sing "Beat It".

   Elicit the fact that, as in 2c and 2e, the negation of a sentence that contains a negative word is the same sentence with the negative word removed.

3. Ask pivotal question: How do we translate a verbal sentence into a mathematical sentence?
   Elicit the procedure of replacing words by symbols. Introduce the symbol $\sim$ for negation. Elicit that symbols may be letters. For example, 3n stands for three times a number.
   In logic, we often use letters to stand for statements. A statement such as "It is raining." may be assigned the letter p. (Note: If more than one statement is used; the letters q, r, s and t are the subsequent letters that can be used.)
   Ask if p is "It is raining.", what is the symbol for "It is not raining."? Elicit $\sim p$. Ask, "What is the negation of $\sim p$?" ($\sim(\sim p) = p$)

4. Introduce the idea of a truth table. Inform the students that in logic we use truth tables to help determine and indicate the truth or falsity of a statement. The truth table will have columns and rows. The truth table for negation will begin as follows:

p	$\sim p$

Elicit that there are only two possibilities for the statement p. That is, p may be true (T) or it may be false (F). For statements, "maybe" is not allowed. Hence the truth table will continue as follows:

p	$\sim p$
T	
F	

Ask pivotal question: What generalization can be made about a statement and its negation?
Elicit: Both cannot be true at the same time. Then complete the truth table.

p	$\sim p$
T	F
F	T

5. Present the statement, (1) "She sits in the first seat."
   a) Elicit that it is an open sentence.
   b) Elicit the negation: "She does not sit in the first seat."
   c) Using the class as the domain, elicit the solution set.
   d) Repeat with each of the following.
      (2) "He has blonde hair."
      (3) "She is not wearing a skirt."
      (4) "He is wearing a hat."
6. Translate each of the above into symbols and then form the negation of each.
      (1) p ($\sim p$)
      (2) p ($\sim p$)
      (3) $\sim p$ (p)
      (4) p ($\sim p$)

   Elicit the concept that the negation of a negation of a sentence is the original sentence.

APPLICATIONS:

1. Give the negation of each of the following statements and determine its truth or falsity.
   - a) $5 = 3 + 2$
   - b) April comes after March.
   - c) $4 \cdot 2 > 9$
   - d) The Jets are a great football team.
   - e) Tom Selleck is not handsome.

2. If p represents "It is raining.",
   q represents "$8 \cdot 3 = 25$" and
   r represents "I love Mr. T.",
   write the negation of each of the following in symbolic form.
   - a) It is raining.
   - b) $8 \cdot 3 \neq 25$
   - c) I do not love Mr. T.
   - d) It is not raining.
   - e) I love Mr. T.
   - f) $8 \cdot 3 = 25$

3. Complete each of the following truth tables.

a)

p	$\sim p$
T	
F	

b)

q	$\sim q$
T	
F	

c)

r	$\sim r$

d)

p	$\sim p$	$\sim(\sim p)$

e)

s	$\sim s$	$\sim(\sim s)$	$\sim(\sim(\sim s)$

4. For each of the following: (1) identify the variable and replace the variable in the original sentence to create a true statement; (2) write the negation of the sentence.
   - a) $x + 4 = 6$
   - b) $3x = 12$
   - c) He is not a basketball star.
   - d) She has curly hair.
   - e) It is not a planet.
   - f) It is a month with 31 days.

SUMMARY:

1. What is the symbol for negation?
2. What is the relationship between a statement and its negation?
3. What is the purpose of a truth table?
4. Give an example of a statement whose negation is true.

HOMEWORK:

1. If the domain of each variable is the set of whole numbers less than 10, find the solution set for each of the following.
   - a) e is an even number.
   - b) e is not an even number.
   - c) $x^2 < 9$
   - d) $x^2 > 9$
   - e) $x^2 = 9$
   - f) p is a prime number.
   - g) p is not a prime number.
   - h) c is a composite number.
   - i) $x + x = 2x$
   - j) $\frac{n}{n} \neq 1$
   - k) $2x + 3 = 12$
2. Negate each of the following.
   - a) 1 is a prime number.
   - b) $\frac{0}{0} = 1$
   - c) 0 is an even integer.
   - d) $x > 1$

e) $y \geq 1$

f) 3 is a prime factor of 10.

g) $\pi = \frac{22}{7}$

h) $\pi = 3.1416$

i) $\pi \neq 3.14$ to the nearest hundredth.

j) The formula for the area of a circle is: $A = \pi D$.

k) All three angles of a triangle measure 60°.

3. Fill in the negation truth table for the statement v.

v	$\sim v$	$\sim(\sim v)$

4. Let the variables represent the statements as indicated.

a) r = "The sun is shining."

b) s = "The color of milk is green."

c) t = "$\pi = 3\frac{1}{7}$."

d) u = "$3 + 4 < 5 - 2$"

State the negations:

$\sim r$, $\sim s$, $\sim t$, $\sim u$

Lesson 3 - Conjunctions

AIM: What is a conjunction?

PERFORMANCE OBJECTIVES: The students will be able to...

1. Describe a conjunction.
2. Translate sentences using the word "and" into the notation of logic using the symbol $\wedge$.
3. Determine, given an open sentence in symbolic notation using $\wedge$, whether the sentence that results from substituting given sentences for the variables is true, false or open.
4. Construct the truth table for conjunction.

VOCABULARY: conjunction, compound sentence

MOTIVATION:

Determine whether each statement is true or false:
- a) The day after Wednesday is Thursday and San Francisco is a city in California.
- b) 6 is an even number and 8 is an odd number.

DEVELOPMENT (Include pivotal questions):

1. Pivotal questions:
    - a) What do these sentences have in common?
    - b) How are these sentences different from those we examined when discussing negations?

   Elicit that these are compound sentences.
   Ask: What is the key word in these sentences? (and)
   Define these statements to be conjunctions in logic.
   Elicit the definition: A conjunction is a sentence formed by two statements joined into one sentence with the word and.

2. Discuss the motivation exercises and elicit that the first sentence is true but the second is false. Elicit that each sentence contains two statements. In the first sentence each of the statements is true; therefore, the sentence is true. In the second sentence, the first statement is true but the second is false; hence, the statement is false.

3. Ask: What do we need to express a conjunction in symbolic form?
   Elicit that there is a need for two (2) letters and a symbol.
   Elicit: We used "p" in a negation. "p" and "q" will be used in a conjunction. Indicate that the symbol for "and" is "$\wedge$" and it is used as follows: $p \wedge q$.
   Let p represent "3 + 4 = 7". Let q represent "Monday begins with an M." Let r represent "Snow is white." Have the students represent the following sentences symbolically.
   a) $3 + 4 = 7$ and Monday begins with an M.
   b) $3 + 4 = 7$ and snow is white.
   c) Snow is white and Monday begins with an M.
   d) $3 + 4 = 7$ and snow is not white.
   e) Monday doesn't begin with an M and $3 + 4 \neq 7$.

4. Elicit a truth table for conjunction will have three (3) columns.

p	q	$p \wedge q$

In constructing this truth table, we must consider all possibilities. Elicit that as each statement can be true or false, there are four possibilities to consider. Hence the truth table starts as follows.

p	q	$p \wedge q$
T	T	
T	F	
F	T	
F	F	

From this we elicit the complete table:

p	q	$p \wedge q$
T	T	T
T	F	F
F	T	F
F	F	F

5. If p represents "Spring has 4 letters.", q represents "8 ÷ 4 = 2", and r represents "He is 6 feet tall.", elicit the sentences that correspond to each of the following and discuss whether each is true, false or an open sentence.
   - a) $p \wedge q$
   - b) $q \wedge r$
   - c) $r \wedge p$
   - d) $\sim p \wedge q$
   - e) $r \wedge \sim p$

   Have students note that only those conjunction sentences in which both parts are true are true sentences.

6. Ask, as a medial summary: When is a conjunction a true statement?

APPLICATIONS:

Use the following sentences for questions 1 and 2.

Let p, q, r and s be as follows.

p: Roses are red.

q: 16 > 15

r: 25 is an even number.

s: Dave Winfield plays first base.

1. Using the above, translate each into a sentence and determine whether it is true or false.
   - a) $p \wedge q$
   - b) $q \wedge r$
   - c) $p \wedge s$
   - d) $q \wedge \sim r$
   - e) $q \wedge \sim s$
   - f) $\sim p \wedge q$

2. Using the above, translate each sentence into symbolic form.
   - a) Roses are red and Dave Winfield plays first base.
   - b) 16 > 15 and roses are red.
   - c) 16 > 15 and 25 is an even number.
   - d) Roses are not red and 16 > 15.
   - e) 25 is not an even number and Dave Winfield doesn't play first base.
   - f) 25 is an even number and 16 is less than 15.

3. Complete the following truth tables.

a)

p	q	$p \wedge q$

b)

r	s	$r \wedge s$

c)

p	q	$\sim p$	$q \wedge \sim p$

d) Construct a truth table for $(\sim q) \wedge (\sim p)$

SUMMARY:

1. What is a conjunction?
2. When will a conjunction be a true statement?
3. How many possibilities must be considered when constructing a truth table for a conjunction?
4. Give an example of a conjunction which is always false. $(p \wedge \sim p)$

HOMEWORK:

1. If the domain of each variable is the set of rational numbers (numbers expressable as the quotient of two integers), find the solution set for each of the following.
   - a) $2x + 3 = 12$
   - b) $2x - 3 = 12$

*c) $2x + 3 = -12$

*d) $2x - 3 = -12$

e) $2x + 3x = 12$

*f) $2x - 3x = 12$

g) $n^2 = 9$

*h) $|y| < 4$

*i) $z^2 = 2$

Note: $|n| = \begin{cases} n \text{ if } n \geq 0 \\ -n \text{ if } n < 0 \end{cases}$, i.e. $|3| = |-3| = 3$

2. Let the variables represent the statements as shown.

p = "The Yankees were a great team."

q = "Swimming is an excellent sport."

r = "A woman was never president of the U.S."

$s = (2.5)^2 < 6.5$

Express in words and state the truth value.

a) $p \wedge q$

b) $p \wedge \sim q$

c) $r \wedge s$

d) $\sim q \wedge \sim r$

e) $\sim p \wedge \sim s$

f) $s \wedge \sim r$

q) $p \wedge s$

h) $\sim p \wedge s$

3. Fill in the truth table.

a)

q	r	$q \wedge r$	$\sim(q \wedge r)$
T			
F			
T			
F			

b)

q	r	$\sim q$	$\sim p$	$\sim q \wedge \sim s$
T				
F				
T				
F				

4. Do the following have the same truth value? If not, show by example.

a) p , $\sim(\sim p)$

b) $\sim p \wedge q$, $p \wedge \sim q$

c) $\sim(p \wedge q)$, $\sim p \wedge \sim q$

d) $p \wedge q$, $q \wedge p$

5. Is conjunction commutative?

6. Does negation distribute over conjunction? (In other words, does $\sim(p \wedge q)$ have the same truth value as $(\sim p \wedge \sim q)$ for all possible truth values of p and q?)

Lesson 4 - Disjunctions

AIM: What is a disjunction?

PERFORMANCE OBJECTIVES: The students will be able to...

1. Describe a disjunction.
2. Translate sentences using the word "or" into the notation of logic using the symbol $\vee$.
3. Determine, given an open sentence in symbolic notation using $\vee$, whether the sentence that results from substituting given sentences for the statement variable is true, false or open.
4. Construct the truth table for disjunction.

VOCABULARY: disjunction, compound statement, truth value

MOTIVATION:

There are two clubs in the school with the following requirements for membership:

HONOR SOCIETY	SERVICE CLUB
85% average	service to the school
and	or
service to the school	service to the community

George has a 78% average, and he has been a monitor in the school library for two years. Is he eligible to join these clubs?

DEVELOPMENT (Include pivotal questions):

1. Elicit that George is eligible to join the Service Club but not the Honor Society. The requirements for the Honor Society involve a conjunction; the requirements for the Service Club involve another connective, namely "or".

Elicit from the students some simple sentences containing "or". Name these sentences "disjunctions" and elicit the following definition:

> A disjunction is a compound sentence in which two statements are joined by the word "or".

Discuss the truth or falsity of each part of the sentences and then the entire sentences.

For example:

a) April has snow or "May" contains three letters.
b) $5 + 7 = 12$ or $5 + 7 = 13$.
c) This teacher is great or Michael Jackson is a singer.

Elicit that if either part is true, then the entire sentence is true.

2. Pivotal question: If the symbol for disjunction is $\vee$, how would a disjunction be represented symbolically?

   Elicit $p \vee q$.

   Using p, q, r and s, have the students represent each of the following symbolically and determine the truth or falsity of the compound sentence.

   p: "John" is a boy's name.
   q: $6 < 12$
   r: 14 is an odd number.
   s: All apples are red.

   a) "John" is a boy's name or $6 < 12$.
   b) 14 is an odd number or all apples are red.
   c) $6 < 12$ or all apples are red.
   d) "John" is a boys name or 14 is not an odd number.

3. The teacher should then reverse the process.

   Present the following symbolic sentences and have the students translate them into words. (Use the same statements.)

   a) $p \vee q$
   b) $p \vee r$
   c) $r \vee q$
   d) $r \vee s$
   e) $p \vee \sim q$
   f) $\sim r \vee \sim s$

Ask, as a medial summary: When is a disjunction a true statement?

4. Develop a truth table. Have the students recall the truth table for conjunction. elicit all the entries.

p	q	$p \wedge q$
T	T	T
T	F	F
F	T	F
F	F	F

From this, elicit that for disjunction we must consider the same possibilities for p and q.

Ask pivotal question: What, if anything, will be different in the disjunction truth table?

The students should note that only the last column is different and should read $p \vee q$. Elicit the table.

p	q	$p \vee q$
T	T	T
T	F	T
F	T	T
F	F	F

Have the students note the difference in the entries in the last column.

Ask: When is a disjunction true? (When at least one statement is true.)

When is a disjunction false? (When both statements are false.)

Combine the truth table for $r \vee s$ and $r \wedge s$ in one as follows.

Ask: How many columns are needed? (4)

Elicit the headings and write the truth table on the board. Elicit all the entries as indicated.

r	s	$r \vee s$	$r \wedge s$
T	T	T	T
T	F	T	F
F	T	T	F
F	F	F	F

5. Present the sentence: He is seven feet tall or sits in the first row.

Ask: Can we determine the truth for this sentence? Elicit that this is an open sentence. Replace the "he" with different names and elicit that the sentences now become disjunctions whose truth value can be determined. Present the following sentences and ask the class whether they are true, false or open.

Use p: It is red. q: $6 + 4 = 8$
r: $x + 5 = 2$ s: She has long hair.

a) $p \vee q$ b) $p \vee r$ c) $r \vee s$
d) $q \vee r$ e) $p \vee s$

Elicit the variables in each of the above.

Again ask: When is a disjunction a true statement?

APPLICATIONS:

Use the statements p, q, r and s as follows for the exercises in part 1 and 2.

p: Basketballs are round. q: Horses have three legs.
r: $18 = 24 - 6$ s: Falling snow is green.

1. Using the statements above, write each of the following in symbolic form and determine the truth value of each sentence.
   - a) Basketballs are round or horses have three legs.
   - b) Horses have three legs or falling snow is green.
   - c) Falling snow is green or $18 = 24 - 6$.
   - d) Falling snow is green or basketballs are round.
   - e) Basketballs are not round or horses have three legs.
   - f) Falling snow is not green or $18 \neq 24 - 6$.
   - g) Horses have three legs or $18 \neq 24 - 6$.

2. Using the statements p, q, r and s above, write each of the following in words and determine its truth value.

a) $p \vee q$ b) $p \vee r$
c) $r \vee s$ d) $q \vee s$
e) $q \vee \sim r$ f) $r \vee \sim p$
g) $q \vee \sim s$ h) $s \vee \sim r$

3. Complete the following truth tables.

a)

p	q	$p \vee q$
T		
T		
F		
F		

p	q	$\sim p$	$q \vee (\sim p)$

b)

r	s	$r \vee s$	$r \wedge s$
T			
T			
	T		
	F		

p	q	$\sim p$	$\sim q$	$(\sim p) \vee (\sim q)$

SUMMARY:

1. What is a disjunction?
2. When is a disjunction a true statement?
3. What is the symbol for disjunction?
4. Give an example of a disjunction which is always true. $(p \vee \sim p)$

HOMEWORK:

1. Let each variable have the set of whole numbers for its domain. Find the solution set for each of the following sentences.
   - a) $(x>4) \wedge (x<10)$
   - b) $(y \text{ is odd}) \wedge (y \text{ is a multiple of } 3) \wedge (y<20)$
   - c) $(z \text{ is even}) \wedge (z \text{ is a multiple of } 3) \wedge (z<20)$
   - d) $(w<4) \vee (w<10)$
   - e) $(u<4) \vee (u+u=10)$
   - f) $\sim(n>4) \vee (n+2=12)$

*2. (For students who know what integers are.) Let each variable have the set of integers for its domain. Find the solution set for each of the following statements.
   - a) $(x<4) \wedge (x^2=25)$

b) $(y+1=7) \vee (2y-3=15)$

c) $\sim(x<4) \wedge (x^2=25)$

d) $(w>-3) \wedge (w<3)$

e) $\sim(w>-3) \wedge \sim(w<3)$

f) $[(w>-5) \wedge (w<5)] \wedge (w^2>1)$

3. Complete the truth table.

a)

p	q	$\sim p$	$\sim q$	$\sim p \wedge \sim q$	$p \vee q$	$\sim(p \vee q)$
T						
F						
T						
F						

b)

p	q	$\sim p$	$\sim q$	$p \vee q$	$p \wedge q$	$\sim(p \wedge q)$
T						
F						
T						
F						

c) Try to express $p \vee q$ in terms of p, q and the connectives $\wedge$ and $\sim$. (Hint: see 3a)

d) Try to express $p \wedge q$ in terms of p, q and the connectives $\vee$ and $\sim$. (Hint: see 3b)

4. Prepare a truth table for

a) $p \vee q$

b) $\sim(p \wedge q)$

5. Prepare a table containing both $p \wedge (q \vee r)$ and $(p \wedge q) \vee (p \wedge r)$ as column headings. What does this table say about $\wedge$ distributing over $\vee$?

6. By means of a truth table decide whether $\vee$ distributes over $\wedge$. (See question 5.)

Lesson 5 - Conditionals

AIM: What is a conditional?

PERFORMANCE OBJECTIVES: The students will be able to...

1. Describe:
   a) conditional
   b) antecedent or hypothesis
   c) consequent or conclusion
2. Given various conditional statements, identify the hypothesis and conclusion.
3. Translate sentences using the words "if..., then..." into notation of logic using the symbol "——>".
4. Translate into words a sentence written in logic notation using any of the connectives ~, ∧ , V, ——>.
5. Construct the truth table for conditionals.
6. Construct the truth table for any open sentence in symbolic form with two variables.

VOCABULARY: conditional, antecedent, hypothesis, consequent, conclusion, implication

MOTIVATION: Write three sentences each beginning with the word "If". What other word is present in each of these sentences? (then)

DEVELOPMENT:

1. Elicit sentences such as: If it is raining, then the streets are wet. Elicit that this is a different kind of compound sentence. It is called a conditional. Elicit the definition and write it on the board.
   Definition: A conditional is a compound sentence of the form "if p then q" where p and q are statements. These

are also called *implications*. Inform the students that each part has a name; the "if" part is called the *antecedent* or *hypothesis*, the "then" part is called the *consequent* or *conclusion*.

2. Present the following conditionals and have the students identify the hypothesis and the conclusion.
   a) If 5 > 4, then it is odd.
   b) If the sun is shining, then it is warm.
   c) Today is Thursday if yesterday was Wednesday.
   d) A figure is a square if it has four sides.
   e) If it is daytime, the sun is shining.

   Elicit that the hypothesis does not have to be at the beginning of a sentence, but it is the part that follows the word "if".

3. Indicate that, as with other terms in logic, the conditional has a symbol, "——>". That is, "p——>q" is read "if p then q".
   Using p, q, r and s as follows, have the students translate the sentences into symbols and those in symbolic form into sentences.

   p: Birds fly south.
   q: It is winter.
   r: New York is a cold city.
   s: Turkeys eat lions.

   a) If birds fly south, then it is winter.
   b) If it is winter, turkeys eat lions.
   c) New York is a cold city if it is winter.
   d) If New York is a cold city, then birds fly south.
   e) p——>q
   f) q——>r
   g) s——>q
   h) r——>p

4. Develop the truth table for the conditional, using a given p and q such as
   p: It is sunny.
   q: We will go on a picnic.

   Elicit again that four possibilities exist.

That is,

p	q
T	T
T	F
F	T
F	F

and we must determine the four truth values of p———>q.

In this case, p———>q (read, "if p then q") would be, "If it is sunny, then we will go on a picnic." This statement may be viewed as a promise. If it is sunny, the promise will be kept (true) if the picnic is held, and the promise will be broken (false) if the picnic is not held. The promise does not predict what will happen if the day is not sunny: whether a picnic is held or not, the promise has not been broken, so the conditional statement remains "true". This analysis permits us to complete the table. The conditional is true except in the case when the hypothesis is true and the conclusion is false.

We want to be able to assign a truth value to the sentence "p ———> q" even when statements p and q are unrelated. The following approach may be used to develop a definition of p ———> q in terms of earlier in for mativate, and so obtain its truth table.

To judge if the conditional "p———>q" is true or false, we say that if the premise, p, is true they the only way a false conclusion, q, can be reached is if p———q is an invalid or false implication. On the other hand, if the premise, p, is false then any conclusion can be reached logically and correctly (i.e., anything can be proved if you start with a false premise); thus "p———q" may be judged a valid or true implication. In summary, a false implication only occurs when a true premise leads to a false conclusion. The following truth table illustrates this point:

p	q	$p \longrightarrow q$
T	T	T
T	F	F
F	T	T
F	F	T

5. More involved truth tables should now be shown. Use the statement $(pVq) \longrightarrow \sim p$. Elicit that five columns will be needed with the third and fourth columns used to determine the truth values for $(pVq) \longrightarrow (\sim p)$.
   Elicit the headings:

p	q	pVq	$\sim p$	$(pVq) \longrightarrow \sim p$

   Then elicit the first four columns.

p	q	pVq	$\sim p$	$(pVq) \longrightarrow \sim p$
T	T	T	F	
T	F	T	F	
F	T	T	T	
F	F	F	T	

   From this the final column is elicited as F
   F
   T
   T

6. Elicit a medial summary: A conditional, $p \longrightarrow q$, is only false if p is true and q is false. (If the hypothesis is true and the conclusion is false, then the conditional is false.)

APPLICATIONS:

1. In each of the following conditionals, identify the hypothesis (antecedent) and the conclusion (consequent).
   a) If Dr. J scores 30 points, then the Sixers win the game.

b) If it is raining, then my mother drives me to school.

c) If she is gorgeous, her name is Jennifer.

d) It is a triangle if it has three sides.

e) Mary is my aunt if I am her nephew.

2. Use p, q, r and s as follows.

p: Donkeys fly.
q: "Monday" begins with an M.
r: $14 > 7$
s: 12 is an odd number.

a) Use the above to translate into symbols and determine the truth or falsity of each.

1. If donkeys fly, then Monday begins with an M.
2. If $14 > 11$, then 12 is an odd number.
3. If 12 is an odd number, then donkeys fly.
4. $14 > 11$ if "Monday" begins with an M.
5. 12 is an odd number if donkeys fly.
6. If $14 > 11$, then donkeys fly.

b) Use the above to translate into sentences. Determine the truth value of each.

1. $p \longrightarrow q$
2. $p \longrightarrow r$
3. $r \longrightarrow s$
4. $s \longrightarrow p$
5. $S \longrightarrow (\sim p)$
6. $q \longrightarrow (\sim s)$
7. $(p \vee q) \longrightarrow r$
8. $q \longrightarrow (r \vee s)$
9. $\sim p \longrightarrow (r \wedge s)$
10. $\sim q \longrightarrow (\sim p \wedge r)$

3. Complete the following truth tables.

a)

p	q	$p \longrightarrow q$

b)

r	s	$r \longrightarrow s$

c)

p	q	$\sim q$	$p \longrightarrow (\sim q)$

d)

p	q	$\sim p$	$p \vee q$	$(\sim p) \longrightarrow (p \vee q)$

SUMMARY:

1. What is a conditional?
2. Which part of the conditional is the hypothesis?
3. When is a conditional true? When is it false?

HOMEWORK:

1. Identify the antecedent and the consequent for each of the following conditionals.
   a) If a triangle is isosceles then it has two congruent angles.
   b) If two angles are right angles, then the two angles are congruent.
   c) If $x = 3$, then $x < 10$.
   d) $c^2 < 100$ if $c < 10$.
   e) If $a + b = 6$, then $c = 7$.
   f) The areas of the two triangles are equal if the triangles are congruent.

2. Prepare a truth table for each of the following.
   a) $u \longrightarrow v$
   b) $\sim u \vee v$
   c) $p \longrightarrow p$
   d) $p \longrightarrow \sim p$
   e) $\sim p \longrightarrow p$
   f) $(p \wedge q) \longrightarrow p$
   g) $(p \vee g) \longrightarrow p$
   h) $p \longrightarrow (p \vee q)$

*3. If the domain of each variable is the set of whole numbers less than ten, find the solution set for each of the following.
   a) $(x<2) \longrightarrow (x<7)$
   b) $(x>2) \longrightarrow (x>7)$
   c) $(x>2) \longrightarrow (x<7)$
   d) $(x<2) \longrightarrow (x>7)$
   e) $(a>3) \longrightarrow (a^2>20)$
   f) $(b<5) \longrightarrow (b^2>10)$
   g) $(c=2) \longrightarrow (c^2>10)$
   h) $(c^2>10) \longrightarrow (c=5)$

*4. Each variable has for its domain TRUE,FALSE . What is the solution set for:
   a) $p \longrightarrow p$
   b) $p \longrightarrow \sim p$
   c) $p \longrightarrow q$
   d) $\sim (p \longrightarrow q)$
   e) $(p \wedge q) \longrightarrow (p \vee q)$
   f) $(p \vee q) \longrightarrow (p \wedge q)$

Lesson 6 - Tautologies

AIM: What is a tautology?

PERFORMANCE OBJECTIVES: The students will be able to...

1. Describe a tautology.
2. Determine, by creating a truth table, whether a sentence is a tautology (two variables only).

VOCABULARY: tautology, contradiction

MOTIVATION: Consider these compound sentences:

1. $p \vee (\sim p)$
2. $p \wedge (\sim p)$
3. $p \longrightarrow (\sim p)$

Are any always true? (Justify with truth tables.)

DEVELOPMENT (Include pivotal questions):

1. Have students write the three truth tables on the board.

a)

p	$\sim p$	$p \vee (\sim p)$
T	F	T
F	T	T

b)

p	$\sim p$	$p \wedge (\sim p)$
T	F	F
F	T	F

c)

p	$\sim p$	$p \longrightarrow (\sim p)$
T	F	F
F	T	T

Inform the students that the first statement is a tautology because of the results of the last column of the truth table. Elicit the definition: A tautology is a compound sentence that is always true.

2. Ask: What could be said about the statement $p \wedge (\sim p)$? Elicit that this statement is always false. State that this is called a contradiction.
Elicit and write the definition: A contradiction is a compound sentence that is always false.

3. Pivotal question: Is $(p\wedge q) \longrightarrow (p\vee q)$ a tautology?
   Elicit that to determine this we must construct the truth table for $(p\wedge q) \longrightarrow (p\vee q)$.
   Ask: What headings are needed?

p	q	$p\wedge q$	$p\vee q$	$(p\wedge q) \longrightarrow (p\vee q)$

   Elicit that in constructing a truth table the first two columns will be for the letters involved, the last column is that statement which is in question, and the other columns are the items necessary to determine the truth of the last statement. Now elicit the entries in the table.

p	q	$p\wedge q$	$p\vee q$	$(p\wedge q) \longrightarrow (p\vee q)$
T	T	T	T	T
T	F	F	T	T
F	T	F	T	T
F	F	F	F	T

   Therefore $(p\wedge q) \longrightarrow (p\vee q)$ is a tautology since it is always true.

APPLICATIONS:

1. Show that each of the following is a tautology.
   a) $p \longrightarrow p\vee q$
   b) $(p\wedge q) \longrightarrow q$
   c) $(\sim p\vee q) \longrightarrow (p \longrightarrow q)$
   d) *Challenge problem $[(p \longrightarrow q)\wedge \sim q] \longrightarrow \sim p$
2. Determine whether each of the following is a tautology, a contradiction or neither.
   a) $\sim s\wedge s$
   b) $\sim q \longrightarrow \sim p$
   c) $(p\vee q) \longrightarrow (p\wedge q)$
   d) $\sim p \longrightarrow \sim(p\vee q)$

SUMMARY:

1. What is a tautology? Illustrate with an English sentence.
2. What is a contradiction? Illustrate with an English sentence.
3. How do we determine if a statement is a tautology or a contradiction?

HOMEWORK:

1. If t and t' represent true statements while f represents a false statement, what is the truth value of each of the following?
   - a) $t \rightarrow f$
   - b) $t \rightarrow t'$
   - c) $(t \wedge f) \rightarrow t'$
   - d) $(t \vee f) \rightarrow t'$
   - e) $\sim t \rightarrow f$
   - f) $(t \vee f) \wedge t'$
   - g) $t \wedge (t' \vee f)$
   - h) $(t \wedge t') \vee (t \wedge f)$

2. Prove that each of the following is a tautology.
   - a) $r \vee \sim r$
   - b) $(r \wedge \sim r) \rightarrow p$
   - c) $[(r \rightarrow s) \wedge (s \rightarrow r)] \rightarrow (\sim r \rightarrow s)$
   - d) $[(r \wedge p) \rightarrow (r \vee p)] \rightarrow (\sim p \rightarrow r)$
   - e) $[(r \vee s) \wedge \sim r] \rightarrow s$

3. Decide whether each of the following is a tautology, a contradiction or neither.
   - a) $\sim p \rightarrow q$
   - b) $q \vee \sim q$
   - c) $q \wedge \sim q$
   - d) $(r \rightarrow s) \rightarrow (s \rightarrow r)$
   - e) $(r \rightarrow s) \rightarrow (\sim s \rightarrow \sim r)$
   - f) $(r \rightarrow s) \rightarrow (r \vee s)$
   - g) $(r \rightarrow s) \rightarrow (r \wedge s)$
   - h) $(r \rightarrow s) \rightarrow (r \vee \sim s)$

4. For each sentence, specify a set of truth values of the statement variables which yield a false statement.
   - a) $p \rightarrow q$
   - b) $p \wedge q$
   - c) $\sim p \wedge \sim q$
   - d) $\sim (p \vee q)$
   - e) $(r \rightarrow s) \rightarrow (s \rightarrow r)$
   - f) $(r \rightarrow s) \rightarrow (\sim s \rightarrow \sim r)$
   - g) $(r \rightarrow s) \rightarrow (\sim r \rightarrow \sim s)$
   - h) $(p \vee r) \rightarrow (\sim p \wedge r)$

Lesson 7 - Truth Tables Using Three Variables

AIM: How many possibilities exist if we were to construct a truth table for three variables?

PERFORMANCE OBJECTIVES: The students will be able to...

1. Construct truth tables given an open sentence in symbolic form with at most three variables.
2. Determine whether a three variable sentence is a tautology.

MOTIVATION:

1. Translate this sentence into symbolic notation, "If a student does service for the school or for the community, then he is eligible for membership in the service club."
2. Make a truth table for this sentence.

DEVELOPMENT:

1. Elicit that since we need a "true" and a "false" for each variable, there are four possibilities for two variables. Hence, there are eight possibilities for three variables as follows.

p	q	r
T	T	T
T	T	F
T	F	T
T	F	F
F	T	T
F	T	F
F	F	T
F	F	F

Note: Each line of a truth table for two variables becomes two lines in a truth table for three variables.

2. From this we could develop any truth table involving three variables. Develop the truth table for (pVq)——>r.

Elicit that this table will have five headings as follows.

p	q	r	pVq	(pVq)———>r

Elicit the table:

p	q	r	pVq	(pVq)———>r
T	T	T	T	T
T	T	F	T	F
T	F	T	T	T
T	F	F	T	F
F	T	T	T	T
F	T	F	T	F
F	F	T	F	T
F	F	F	F	T

Ask: Is this a tautology? (No - only some students in the school can be members.)

3. Ask: Is the statement (pVq)Vr a tautology?
Elicit that to determine this we must develop a truth table and inspect the last column.
Note: The first four columns from the above truth table could be used. the final column would be headed (pVq)Vr and would be:

(pVq)Vr
T
T
T
T
T
T
T
F

APPLICATIONS:

1. Write a truth table for each of the following.
   - a) ∼p———>(qVr)
   - b) pV(q∧r)
   - c) (pVr)———>q
   - d) (sVr)———>t
   - e) (p———>q)∧(q———>r)
   - f) [(p——>q)∧(q——>r)]——>(p——>r)

2. For each of the statements above, determine whether it is a tautology, a contradiction or neither.

SUMMARY:

1. When constructing a truth table for three variables, how many possibilities must we consider?
2. What are the entries in the first three columns of a three variable truth table?
3. How many rows would be needed for a truth table for a sentence involving four variables?

HOMEWORK:

1. Establish that the following are tautologies.
   a) $[(p \vee q) \vee r] \wedge [\sim p \wedge \sim q] \longrightarrow r$
   b) $[p \longrightarrow (r \vee s)] \longrightarrow [(p \longrightarrow r) \vee (p \longrightarrow s)]$
   c) $[p \wedge (q \vee r)] \longrightarrow [(p \wedge q) \vee (p \wedge r)]$
   d) $[p \vee (q \wedge r)] \longrightarrow [(p \vee q) \wedge (p \vee r)]$

2. For what truth values of the statement variables are the following false?

a)	$(r \wedge s) \longrightarrow p$	f)	$p \wedge (r \vee s)$
b)	$(r \vee s) \longrightarrow p$	g)	$p \vee (r \vee s)$
c)	$p \longrightarrow (r \wedge s)$	h)	$(p \longrightarrow q) \wedge r$
d)	$p \longrightarrow (r \vee s)$	j)	$(p \longrightarrow q) \wedge (q \longrightarrow p)$
e)	$p \vee (r \wedge s)$	k)	$(p \longrightarrow q) \vee (q \longrightarrow p)$

3. For what values of the variables are the following statements false?

a)	$\frac{x}{x} = 1$	d)	$\frac{ab}{a} = b$
b)	$\frac{a-1}{a-1} = 1$	e)	$z = z + 1$
c)	$\frac{2y}{y} = 2$	f)	$\frac{1}{x} = \frac{1}{x+1}$

4. Decide whether or not the following are tautologies.
   a) $[(p \longrightarrow q) \longrightarrow r] \longrightarrow [p \longrightarrow (q \longrightarrow r)]$
   b) $[p \wedge (q \vee r)] \longrightarrow [(p \wedge q) \vee (p \wedge r)]$
   c) $\sim(p \wedge q) \longrightarrow (\sim p \vee \sim q)$
   d) $\sim(p \wedge q) \longrightarrow (\sim p \wedge \sim q)$

e) $(p \vee q) \longrightarrow (p \wedge q)$

5. For what values of the statement variables are the following false?

a)	$p \longrightarrow q$	e)	$(p \vee q) \vee (r)$
b)	$(p \longrightarrow q) \longrightarrow r$	f)	$(p \wedge q) \longrightarrow r$
c)	$p \longrightarrow (q \longrightarrow r)$	g)	$\sim p \longrightarrow (p \wedge r)$
d)	$(p \wedge q) \wedge (r)$		

*6. The following table will be used for replacing logical symbols with arithmetic symbols.

Logic	Arithmetic
true	1
false	0
$\sim$	Add 1 (Note: 0+1=1, 1+1=0)
$\wedge$	Multiply

The negation and conjunction truth tables become:

p	$\sim p$ $(1+p)$
0	1
1	0

p	q	$p \wedge q$ (pq)
0	0	0
1	0	0
0	1	0
1	1	1

a) What is the corresponding disjunctive table? Show that $p \vee g$ has the value $p+q+pq$. (Remember that $1+1=0$.)

b) What is the corresponding conditional table? Show that $p \longrightarrow q$ has the value $1+p+pq$.

c) What is a simpler expression for
   1) $p + p$ or $2p$
   2) $p \cdot p$ or p^2
   3) $p(p + q) + p$

   (Except for 1+1=0, the usual rules of arithmetic hold.)

d) What arithmetic value will
   1) a tautology have?
   2) a contradiction have?

e) Making the above replacements, show that the following are tautologies.

1) $p \longrightarrow p$
2) $p \vee \sim p$
3) $(p \wedge \sim p) \longrightarrow q$
4) $(p \longrightarrow \sim q) \longrightarrow (q \longrightarrow \sim p)$
5) $\left[(p \vee q) \wedge \sim p\right] \longrightarrow q$

Lesson 8 - Converses and Inverses of Conditionals

AIM:

1. What is the converse of a conditional statement?
2. What is the inverse of a conditional statement?

PERFORMANCE OBJECTIVES: The students will be able to...

1. Describe the converse of a conditional statement.
2. Describe the inverse of a conditional statement.
3. Write the converse of a given conditional statement of the form $p \longrightarrow q$.
4. Write the inverse of a given conditional statement of the form $p \longrightarrow q$.
5. Determine the truth value of the converse and inverse of a given conditional statement whose truth value has been determined.

VOCABULARY: converse, inverse

MOTIVATION:

Let p = "It is a horse."
q = "It is animal."
Write out the sentences for $p \longrightarrow q$, $q \longrightarrow p$ and $\sim p \longrightarrow \sim q$.
Are any of these true?

DEVELOPMENT (Include pivotal questions):

1. Elicit the three sentences from above as:
    1) If it is a horse, then it is an animal. (true)
    2) If is is an animal, then it is a horse. (false)
    3) If it is not a horse, it is not an animal. (false)

Tell the class that statement #2 is called the converse of the conditional statement #1.

Elicit the definition: The converse of a conditional is another conditional in which the hypothesis and conclusion are interchanged. That is, the hypothesis of the statement becomes the conclusion of the converse and vice versa. Symbolically, if the conditional is p———>q, its converse is q———>p.

Ask the class to state the converse of:

  a) If she is gorgeous, then her name is Jennifer.
  b) If he is brilliant, then his name is Stanley.
  c) If donkeys fly, then 8 > 10.

2. Similarly, state that statement #3 is called the inverse of statement #1.

   Elicit the definition: The inverse of a conditional is one that negates both the hypothesis and conclusion of the original conditional.

   Symbolically, if the conditional is p———>q, its inverse is ~p———>~q.

   Using the statements in #1 from the previous page, ask for the inverse of each.

3. Present the conditional r———>s.

   Ask the students to represent both its converse and its inverse.

   Elicit: The converse is s———>r.
   The inverse is ~r———>~s.

   Medial summary: Pivotal question: How do we form the converse and inverse of a conditional?

4. Construct the truth table for the converse and the inverse of a conditional. Elicit the headings.

p	q	~p	~q	p———>q	q———>p	(~p)———>(~q)

   Above the appropriate columns write the words conditional (p———>q), converse (q———>p), and inverse (~p)———>(~q). The truth table should then be elicited.

p	q	$\sim p$	$\sim q$	$p \longrightarrow q$	$q \longrightarrow p$	$(\sim p) \longrightarrow (\sim q)$

5. Discuss the truth and falsity of converses and inverses of statements.
   Pivotal question: If a conditional is true, will its converse be true? ...inverse be true?
   Elicit conditionals and converses (inverses) where the converse(inverse) is true and where it is false.
   a) Statement: If today is Wednesday, then tomorrow is Thursday.
      Converse: If tomorrow is Thursday, then today is Wednesday.
      Inverse: If today is not Wednesday, then tomorrow is not Thursday.
   b) Statement: If a figure is a square, then it has four sides.
      Converse: If it has four sides, then a figure is a square.
      Inverse: If a figure is not a square, then it does not have four sides.
   Elicit: A converse (inverse) of a true statement may be true but it could also be false.

APPLICATIONS:

1. Write in symbols the converse and inverse of each of the following.
   a) $p \longrightarrow q$ b) $t \longrightarrow r$
   c) $s \longrightarrow w$ d) $\sim p \longrightarrow q$
   e) $\sim q \longrightarrow p$ f) $\sim q \longrightarrow \sim p$
2. Write the converse and inverse of each of the following conditionals.
   a) If the sun is shining, it is warm.
   b) If 5 is odd, then today is Monday.
   c) If he sings, his name is Michael Jackson.

d) If $5 > 9$, then $3 > 2$.
e) Tomorrow is Sunday, if today is Saturday.
f) If rain is not wet, then ice is not cold.

3. Construct the truth table for the
    a) Converse of r———>s.
    b) Inverse of r———>s.
    c) The converse and inverse of x———>y.

SUMMARY:

1. What is the converse of a statement?

2. What is the inverse of a statement?

3. Is the converse (inverse) of a true statement true?

4. Give an example of a true statement whose converse is true; a true statement whose converse is false; a false statement whose converse is true, false.

5. Similar to question 4, for inverses.

6. (For any of the statements discussed in this lesson) What is the converse of the converse of this statement?

*7. Will the converse and inverse of a statement always have the same truth value? Can you think of a case where one is true and the other is false?

HOMEWORK:

1. State the converse of each of the following.
    a) If it is water, then it is wet.
    b) If it has six legs, then it is an insect.
    c) You fail the course if you fail the final exam.
    d) It has wings if it flies.

2. In problem 1, state the inverse for each conditional.

3. For each of the following conditionals, state the truth value of the conditional, its converse and its inverse.
    a) If it produces eggs it is a female.
    b) If it has four congruent sides it is a square.
    c) If $x + 3 = 7$, then $x = 4$.

d) If $y > 5$, then $y > 6$ whenever y is a number.

e) If $z^2 = 9$, then $z = 3$ whenever z is an integer.

f) If $1 + 1 = 11$, then $2 + 2 = 22$.

4. Using a truth table, decide whether or not the converse and inverse of the same conditional always have the same truth values.

5. Decide which of the following is a tautology.

   a) $(p \longrightarrow q) \longrightarrow (q \longrightarrow p)$

   b) $(p \longrightarrow q) \longrightarrow (\sim p \longrightarrow \sim q)$

   c) $(q \longrightarrow p) \longrightarrow (\sim p \longrightarrow \sim q)$

   d) $\sim (q \longrightarrow p) \longrightarrow (p \wedge \sim q)$

6. Answer TRUE or FALSE.

   a) If the conditional is true, then the consequent is true.

   b) If the consequent is true, then the conditional is true.

   c) If the antecedent is true, the conditional is true.

   d) If the antecedent is false, then the conditional is true.

7. When is a conditional true and its consequent false?

8. When are both the inverse and the converse of the conditional, $p \longrightarrow q$, true simultaneously?

9. On a test, Peter was aked to find the area of a quadrilateral. He measured one side and multiplied the measurement by itself. Later, his teacher asked him why he had assumed it was a square. He explained, "A square has four sides." What was the error in his reasoning?

10. A student must pass the Math RCT (or a Math Regents) to get a high school diploma. Jack passed the Math RCT, but did not get a diploma. Was he being treated unfairly?

Lesson 9 - Contrapositives of Conditionals

AIM: What is the contrapositive of a conditional statement?

PERFORMANCE OBJECTIVES: The students will be able to...

1. Describe the contrapositive of a conditional statement.
2. Write the contrapositive of a given conditional statement of the form p——>q.
3. Determine the truth value of a contrapositive, converse and inverse of a given conditional statement whose truth value has been determined.
4. Select from several conditional sentences the sentence which is logically equivalent to the given sentence.

VOCABULARY: contrapositive, logically equivalent

MOTIVATION:

Given the conditional: If it is a square, then it has four sides. What is its relationship to the following sentence: If it doesn't have four sides, then it is not a square.

DEVELOPMENT (Include pivotal questions):

1. Ask: Is the second sentence the converse of the conditional? Is it the inverse?
   Elicit that it is more than either of these.
   Review: The converse interchanges the if..., then, and the inverse negates both the hypothesis and conclusion.
   Pivotal question: What changes have taken place between these two conditionals?
   Elicit: The hypothesis and conclusion are interchanged and the new if and then are negated.

Ask: If we used two names for this change, what would it be called?

Elicit: This is the converse of the inverse or the inverse of the converse. This is called the contrapositive of the original conditional.

Elicit the definition: The contrapositive of a conditional is a conditional in which the hypothesis and conclusion are interchanged and negated.

Ask: What is the contrapositive of

a) If it is raining, the streets are wet.
b) If she is adorable, then her name is Yvette.
c) He is tall if he plays basketball.

2. Ask: How do we represent a conditional symbolically?

Elicit: $p \longrightarrow q$

Ask: How would its contrapositive be represented symbolically?

Elicit: $\sim q \longrightarrow \sim p$

Ask: What are the converse, inverse and contrapositive of

a) $r \longrightarrow s$
b) $q \longrightarrow p$
c) $s \longrightarrow t$

Note: Review that the negation of a negation is the original (i.e., $\sim(\sim p)$ is p).

3. Ask: What are the headings of a truth table to determine the truth value of the contrapositive $(\sim q) \longrightarrow (\sim p)$?

Elicit:

p	q	$\sim q$	$\sim p$	$(\sim q) \longrightarrow (\sim p)$

Then elicit the entire truth table:

p	q	$\sim q$	$\sim p$	$(\sim q) \longrightarrow (\sim p)$
T	T	F	F	T
T	F	T	F	F
F	T	F	T	T
F	F	T	T	T

Ask: What are the truth values of p——>q?
Add the column onto the truth table.

p——>q
T
F
T
T

4. Ask: What is the relationship between the truth values of the conditional and the contrapositive?
   Elicit: The truth values are identical.
   Pivotal question: If a conditional is true, will the contrapositive always be true? Similarly, if the conditional is false, will the contrapositive always be false?
   Elicit: The answer to both is yes.
   We therefore say that: The contrapositive and the original conditional are logically equivalent statements.
   Present the statement: If today is Tuesday, then tomorrow is Wednesday.
   Ask: Which of the following are logically equivalent to the above?
   a) If tomorrow is Wednesday, then today is Tuesday.
   b) If tomorrow is not Wednesday, then today is not Tuesday.
   c) If today is not Tuesday, then today is not Wednesday.

   Similarly present the statement: If $5 > 3$, then 3 is odd.
   Ask: Which of the following are logically equivalent?
   a) If 3 is not odd, then $5 > 3$.
   b) If $5 > 3$, then 3 is not odd.
   c) If 3 is odd, then $5 \not> 3$.

5. Construct a truth table for the converse, inverse and contrapositive of p——>q. Elicit the headings and the table.

p	q	$\sim p$	$\sim q$	Statement $p \longrightarrow q$	Converse $q \longrightarrow p$	Inverse $(\sim p) \longrightarrow (\sim q)$	Contrapositive $(\sim q) \longrightarrow (\sim p)$
T	T	F	F	T	T	T	T
T	F	F	T	F	T	T	F
F	T	T	F	T	F	F	T
F	F	T	T	T	T	T	T

Medial Summary: Which of the above are logically equivalent?
The student should observe two logically equivalent statement-pairs: 1) the statement and its contrapositive and 2) its converse and inverse.

APPLICATIONS:

1. Write the contrapositive of each of the following in symbolic form.
   a) $p \longrightarrow q$
   b) $r \longrightarrow s$
   c) $q \longrightarrow r$
   d) $\sim p \longrightarrow q$
   e) $\sim r \longrightarrow \sim s$
   f) $t \longrightarrow \sim r$

2. Write the contrapositive of each of the following sentences.
   a) If a figure has three sides, then it is a triangle.
   b) If it is winter, then it is cold.
   c) If she is terrific, her name is Cheryl.
   d) Cows race if rabbits creep.
   e) If he is strong, then he is Mr. T.

3. Given the following:
   a) If $9 > 8$, today is Monday.
   b) If today is Monday, then tomorrow is Tuesday.
   c) If tomorrow is Tuesday, then $9 > 8$.

   Which of the following are logically equivalent to any of the above statements? Match those which are.
   1) If tomorrow is Tuesday, then today is Monday.
   2) If $9 > 8$, then tomorrow is not Tuesday.
   3) If $9 > 8$, then today is not Monday.
   4) If tomorrow is not Tuesday, then today is not Monday.

5) If today is not Monday, then $9 \not> 8$.

4. Construct truth tables for the converse, inverse and contrapositive of each of the following.
    a) $r \longrightarrow s$
    b) $q \longrightarrow r$
    c) $\sim p \longrightarrow \sim t$
    d) $\sim a \longrightarrow b$

SUMMARY:

1. How is the contrapositive of a statement formed?

2. When are two statements logically equivalent?

HOMEWORK:

1. State the contrapositive of each of the following.
    a) If the integer is one more than some even integer, then it is an odd integer.
    b) If two angles are right angles, then they are congruent.
    c) If the product of two integers is an odd integer, then both factors are odd integers.
    d) If $x = 5$, then $x^2 = 25$.
    e) If $y > 7$, then $y > 2$.
    f) If ΔABC is a right triangle with c being the hypotenuse, then $a^2 + b^2 = c^2$.
    g) If it is a legal holiday, then school is closed.
2. What is the converse and contrapositive of each of the following?
    a) $p \longrightarrow s$
    b) $\sim p \longrightarrow q$
    c) $r \longrightarrow \sim s$
    d) $\sim u \longrightarrow \sim v$
    e) If it is hot, then I perspire.
    f) If I am hungry, then I eat.
    g) If the triangle is equilateral, then it is isoceles.

3. Given the statement:

   If a person drives while intoxicated, he will be sent to jail.

   Which of the following are logically equivalent to it?

   a) If a person does not drive while intoxicated, he will not be sent to jail.
   b) If a person is not sent to jail, then he did not drive while intoxicated.
   c) A person will be sent to jail if he drives while intoxicated.
   d) If a person is sent to jail, then he drove while intoxicated.
   e) If a person drives while intoxicated, he will not be sent to jail.

4. Thinking of the connectives $\vee$, V, $\sim$, $\longrightarrow$ as binary operations on statements, decide whether these have the same truth values, and state the kind of distributivity if any.

   a) $p\wedge(qVs)$, $(p\wedge q)V(p\wedge s)$
   b) $p \longrightarrow (qVs)$, $(p \longrightarrow q)V(p \longrightarrow s)$
   c) $(pVq) \longrightarrow s$, $(pVs) \longrightarrow (pVs)$
   d) $\sim(p\wedge s)$, $(\sim pV\sim s)$

Lesson 10 - Biconditionals

AIM: What is a biconditional statement?

PERFORMANCE OBJECTIVES: The students will be able to...

1. Describe a biconditional statement and define symbolically that p<———>q means (p———>q) and (q———>p).
2. Translate sentences using words "if and only if" and the symbol <———>.
3. Construct a truth table using biconditionals.
4. Determine if statements are logically equivalent.

VOCABULARY: bicondiitonal

MOTIVATION:

Construct a truth table for:

1) If today is Friday, then tomorrow is Saturday.
2) If tomorrow is Saturday, then today is Friday.

DEVELOPMENT (Include pivotal questions):

1. Elicit the headings and truth table as follows.

p	q	p———>q	q———>p
T	T	T	T
T	F	F	T
F	T	T	F
F	F	T	T

Ask: If a conjunction was made of these two conditionals, what would it be?
Elicit: (p———>q)∧(q———>p) is the statement.
Ask: What one word could be used to describe the conjunction of these two conditionals?

Elicit: The prefix "bi" means 2; therefore, biconditional is the term used to describe this statement.

Now add the column for $p \longleftrightarrow q$, eliciting that this means $(p \longrightarrow q) \wedge (q \longrightarrow p)$.

The table will be:

p	q	$p \longrightarrow q$	$q \longrightarrow p$	$p \longleftrightarrow q$
T	T	T	T	T
T	F	F	T	F
F	T	T	F	F
F	F	T	T	T

Elicit: A biconditional is true only when both statements are true or both are false.

2. Present p as "it is raining" and q as "the streets are wet".

   Ask: Write the following in words.

   a) $p \longrightarrow q$: If it is raining, then the streets are wet.

   b) $q \longrightarrow p$: If the streets are wet, then it is raining.

   c) $(p \longrightarrow q) \wedge (q \longrightarrow p)$: If it is raining, then the streets are wet and if the streets are wet, then it is raining.

   Have the class note that this is cumbersome. Tell them we use the words "if and only if" for biconditionals. This is sometimes abbreviated as "iff".

   Elicit the statement for $p \longleftrightarrow q$ as "It is raining if and only if the streets are wet."

   Present p: Donkeys fly.

   q: $6 + 5 = 11$

   r: "Tuesday" begins with a "T"

   Have the students write the following in words.

   1) $p \longleftrightarrow q$
   2) $q \longleftrightarrow r$
   3) $p \longleftrightarrow r$

Construct a truth table for $p \longleftrightarrow q$. Elicit the headings and the table.

p	q	$\sim p$	$(\sim p) \longleftrightarrow q$
T	T	F	F
T	F	F	T
F	T	T	T
F	F	T	F

4. Construct the truth table for $(r \longrightarrow s) \longleftrightarrow (r V s)$. Elicit the headings and the table.

r	s	$\sim r$	$\sim s$	$r \longrightarrow (\sim s)$	$(\sim r) V (\sim s)$	$[r \longrightarrow (\sim s)] \longleftrightarrow [(\sim r) V (\sim s)]$
T	T	F	F	F	F	T
T	F	F	T	T	T	T
F	T	T	F	T	T	T
F	F	T	T	T	T	T

Ask: By looking at the last column, what is the biconditional $[r \longrightarrow (\sim s)] \longleftrightarrow [(\sim r) V (\sim s)]$?

Elicit: This is a tautology.

Ask: When is a biconditional true?

Elicit: A biconditional is true when both statements are true or both are false.

Pivotal question: What conclusion can we draw when a biconditional is a tautology?

Elicit that a biconditional can only be a tautology when the truth values for the two (2) statements involved are identical.

Ask: When the truth values of two statements are identical, what exists?

Elicit: Logical equivalence.

This should lead to the conclusion (elicit and write on board): When a biconditional is a tautology, the two sentences that make up the biconditional are logically equivalent.

APPLICATIONS:

1. Use p, q, r, s as follows:

   p: May has three letters.
   q: Fish fly.
   r: $9 \div 3 > 3$
   s: Chickens lay eggs.

   a) Translate each of the following into words and determine the truth value.

      1) $p \longleftrightarrow q$ 2) $r \longleftrightarrow s$
      3) $p \longleftrightarrow s$ 4) $q \longleftrightarrow r$
      5) $\sim q \longleftrightarrow p$ 6) $\sim r \longleftrightarrow s$

   b) Translate each of the following into symbols.

      1) May has three letters if and only if fish fly.
      2) $9 \div 3 > 3$ if and only if chickens lay eggs.
      3) Fish fly if and only if chickens lay eggs.
      4) Fish don't fly if and only if May has 3 letters.
      5) Chickens don't lay eggs if and only if fish fly.
      6) $9 \div 3 > 3$ if May doesn't have three letters.

2. Construct a truth table for each of the following.

   a) $r \longleftrightarrow s$ b) $q \longleftrightarrow r$
   c) $(\sim p \longrightarrow q) \longleftrightarrow p$

3. Construct the truth table and determine whether the two conditions that make up the biconditional are equivalent.

   a) $\sim(p \vee q) \longleftrightarrow (\sim p \wedge \sim q)$
   b) $(r \longrightarrow s) \longleftrightarrow (\sim r \vee s)$
   c) $(\sim p \longrightarrow q) \longleftrightarrow (p \vee q)$

SUMMARY:

1. What is the symbol for a biconditional?

2. When is a biconditional true?

3. What words indicate that a statement is a biconditional?

4. When does a logical equivalence exist?

HOMEWORK:

1. Prepare a truth table for
   a) $r \longleftrightarrow s$
   b) $\sim(r \wedge s) \longleftrightarrow (\sim r \vee \sim s)$
   c) $\sim(r \vee s) \longleftrightarrow (\sim r \wedge \sim s)$

2. Which of the biconditionals in Example 1 above are logical equivalents?

3. Prove that $\sim(p \longrightarrow q) \longleftrightarrow (p \wedge \sim q)$ is a tautology.

4. If the domain of x is the set of whole numbers less than ten, solve each of the following.
   a) $(x<5) \longleftrightarrow (x=3)$
   b) $(y<5) \longleftrightarrow (y>1)$
   c) $(z<1) \longleftrightarrow (z>5)$

5. For what truth values of the statement variables is the following false?
   a) $r \longrightarrow s$
   b) $p \longleftrightarrow q$
   c) $p \longrightarrow (q \longrightarrow r)$
   d) $(p \longrightarrow q) \longrightarrow r$
   e) $p \vee (q \vee r)$
   f) $(p \wedge q) \wedge r$

6. Try to find a sentence that will fit the last column. Can you find a second sentence that fits?

p	q	? ?	?
T	T		F
T	F		F
F	T		T
F	F		F

7. Fill in the table with YES or NO.

	Connective		
Property	$\vee$	$\longrightarrow$	$\longleftrightarrow$
Commutative			
Associative			
Has an identity			

8. Which of: $\wedge$, $\vee$, $\longrightarrow$, $\longleftrightarrow$ distribute over which? (Note that there are sixteen cases to examine.)

9. Form a biconditional from the statements "today is Friday" and "there is no school tomorrow". Is the biconditional a tautology? Explain.

Lesson 11 - Symbols in Algebra

AIM: What are some of the symbols used in algebra?

PERFORMANCE OBJECTIVES: The students will be able to...

1. Use symbols to represent numbers and operations.
2. List various words used for each operation.

VOCABULARY: sum, difference, quotient, product, greater than, less than, addend, factor, diminished by, total, increased by, decreased by, variable

MOTIVATION:

A student performed four different calculations, using the numbers 9 and 4, and got a different answer each time. What operation did he use to get each of these answers?

a) $2\frac{1}{4}$ c) 5
b) 36 d) 13

DEVELOPMENT (Include pivotal questions):

1. Elicit that the four basic operations of arithmetic are addition, subtraction, multiplication and division.
2. Elicit the standard symbolic notations for each operation:
   a) $9 \div 4$, $\frac{9}{4}$, $9/4$, $4\overline{)9}$
   b) $9 \cdot 4$, $9(4)$, $(9)(4)$
   Note: 9x4 should be discouraged, since the operation sign may be mistaken for a variable.
   c) $9 - 4$
   d) $9 + 4$
3. 9 + 4 may be read as "9 increased by 4". Ask the students to suggest other ways this phrase may be read. List correct responses on the board.

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4. Continue for each operation (what words or phrases could be used to indicate subtraction?) creating a chart similar to the one below to show alternatives.

Addition 9+4	Subtraction 9−4	Multiplication $9 \cdot 4$ 9(4) (9)(4)	Division $9 \div 4$ $\frac{9}{4}$, 9/4 $4\overline{)9}$
9 increased by 4	9 decreased by 4	9 times 4	9 divided by 4
9 plus 4	9 minus 4	the product of 9 and 4	4 divided into 9
the sum of 9 and 4	9 diminished by 4		the quotient of 9 and 4
4 more than 9	4 less than 9		
4 greater than 9	the difference of 9 and 4		
	4 subtracted from 9		

9 + 4 or 13 can also be described as "the number which exceeds 9 by 4". Students should not confuse the phrase "4 more than 9" with the sentence "4 is more than 9". This phrase represents a statement and is false.

5. Students should be made aware that the order of terms in subtraction and division is very important, in both symbolic and English phrases. 9 - 4 is not the same as 4 - 9. Appropriate cautions should be noted on these two columns.

6. Ask: How can we use symbols to represent "the sum of a number and 7"? Elicit that the letters may be used to take the place of unknown numbers. When this is done, we always write a legend or key to explain.
Provide the definition: A variable is a letter used to represent a number.

7. Ask the students to represent "4 more than x", "3 less than n", "m less than q". Explain that the product of a number and a variable (such as 3 times f) should be written without parentheses as 3f; similarly, the product of x and y should be written as xy.

APPLICATION:

1. For each phrase below,
   a) state the operation indicated
   b) write using symbols
      1) the sum of nineteen and eleven
      2) from sixty-three subtract twenty-two
      3) thirty-two decreased by twelve
      4) four less than seven
      5) eleven less than three
2. For each phrase below,
   a) state the operation indicated
   b) write using symbols
      1) the product of a number and eight
      2) twenty-four divided by n
      3) eleven diminished by a number
      4) twice a number
      5) the quotient of twenty-five and x
      6) b greater than 12
      7) one third of a number
      8) eight more than nine times y
      9) x less than the quotient of y by z
3. Translate into English phrases.
   a) 4n
   b) b + 13
   c) x - 5
   d) 3n + 2
   e) 17/m
   f) 30 - x

SUMMARY:

1. Show twelve divided by three in symbols three different ways.

$$\left(\frac{12}{3},\ 12 \div 3,\ 3\overline{)12}\right)$$

2. What is a variable?
3. The order of the terms is especially important for two of the four operations. Which ones?

HOMEWORK:

1. Write the following in symbols.
    a) the sum of sixteen and eleven
    b) twenty-two decreased by nine
    c) the quotient of a number and two
    d) eight times thirty
    e) seventy-one added to a number
    f) ten divided by x
    g) from nineteen subtract n
    h) six increased by a number
    i) the average of a and b
    j) half the product of b and h
    *k) half the product of h and the sum of b and c
2. Write in words:
    a) $4 + 9$
    b) $n - 3$
    c) $5x$
    d) $12 - x$
    e) $12 \div x$

Lesson 12 - Order of Operations

AIM: What is meant by "Order of Operations"?

PERFORMANCE OBJECTIVES: The students will be able to...

1. State the order of operations.
2. Identify grouping symbols.
3. Evaluate numerical expressions using order of operations.

VOCABULARY: order of operations, evaluate, parentheses

MOTIVATION:

1. Find the value of (evaluate):
    - a) 3 + 6 x 2
    - b) 12 ÷ 3 x 2
    - c) 7 - 3 + 2
2. Write in symbols and find the value of the sum of four and nine multiplied by two.

DEVELOPMENT (Include pivotal questions):

1. Discuss answers to motivation 1. Elicit the need for a unique answer. Ask, "In what order should we do the operations in the examples so that we all get the same answer?"
2. Develop conventional agreement:
    - a) We do all the multiplications and divisions first, in the order in which they appear, from left to right.
    - b) Then we do all the additions and subtractions in the order in which they appear, from left to right.

3. Demonstrate that the answers to motivation 1 a, b, c above are 15, 8 and 6, respectively.

4. Do application 1.

5. Discuss motivation 2. How can we show the sum of four and nine is multiplied by two? Develop the need for parentheses to group numerals within an expression. The expression is written (4+9) x 2. Ask, "How do these parentheses affect the order of operations?" Elicit that operations indicated within parentheses are performed before other arithmetic operations. Show that for this problem:

$$\begin{aligned}(4 + 9) \times 2 &= 13 \times 2 \\ &= 26\end{aligned}$$

6. Do applications 2, 3.

APPLICATIONS:

1. Evaluate:
   a) $18 - 6 \div 3$
   b) $38 - 12 \times 3$
   c) $21 - 6 \div 3 + 5 \times 2$
   d) $6 + 4 \times 3 - 12 \div 3$

2. Find the value of:
   a) $55 - (7 - 3)$
   b) $5(3 + 3)(7 - 3)$

3. Write in symbols and evaluate:
   a) Fifteen is subtracted from the product of four and six.
   b) The product of eight and four is added to twenty.

SUMMARY:

1. Which is greater: $(18 - 6) \div 3$ or $18 - (6 \div 3)$?

2. What is the value of $18 - 6 \div 3$?

3. Why must there be a conventional agreement about the order of operations?

HOMEWORK:

1. Find the value of:
   - a) $5 + 1 \times 2$
   - b) $4 \times 3 - 5$
   - c) $(11 - 9) \div 2$
   - d) $25 - 15 \div 3 + 5$
   - e) $32 \div 8 \div 4$
   - f) $32 \div (8 \div 4)$
   - g) $8(7 + 2)(9 - 4)$
   - h) $8(7 + 2)-(9 - 4)$
   - i) $18 \div 2 - (11 - 3)$
   - j) $15 - (6 + 4) \div 2 - 10$
   - k) $[15 - (6 + 4)] \div 2 + 3$
   - l) $(6 + 4)\div(2 + 3) \div 4$

2. Write using symbols and evaluate:
   - a) the sum of eleven and four is divided by the product of five and three
   - b) nine decreased by the product of two and three

*3. For each of the following problems which words might create difficulty for one trying to solve it?
Find their meanings and solve.
   - a) Find the sum of the smallest two positive odd composite numbers.
   - b) Find two different integers whose reciprocals add up to $\frac{1}{2}$.
   - c) Prove that 3.25 is a rational number.

Lesson 13 - Evaluating Algebraic Expressions

AIM: How do we evaluate algebraic expressions?

PERFORMANCE OBJECTIVES: The students will be able to...

1. State what is meant by an algebraic expression.
2. State what is meant by evaluating an algebraic expression.
3. Evaluate algebraic expressions using powers and parentheses.

VOCABULARY: algebraic expression, exponent, factor

MOTIVATION:

1. According to the radio, the temperature is 20°C. What is the temperature in Fahrenheit degrees? (The Fahrenheit temperature is 32° more than nine-fifths of the Centigrade reading.)
2. Write an expression involving the variable c which could be used to find Fahrenheit equivalents for any Centigrade temperature.

DEVELOPMENT (Include pivotal questions):

1. Ask, "What must we know in order to evaluate a phrase which contains a variable?" Show how the value of the expression changes as the value of the variable changes.

In 3n, if n = 1	if n = 5	if n = 100
3n	3n	3n
= 3(1)	= 3(5)	= 3(100)
= 3	= 15	= 300

2. Develop that an algebraic expression is an expression which contains one or more variables.

3. Demonstrate that to find the value of an algebraic expression, we
   - a) replace the variable in the expression with its given value at each occurrence.
   - b) follow the order of operations.

4. Show if $x = 5$,

   a) $3x + 2$
   $= 3(5) + 2$
   $= 15 + 2$
   $= 17$

   b) $3(x + 2)$
   $= 3(5 + 2)$
   $= 3(7)$
   $= 21$

   c) $4x + 2x$
   $= 4(5) + 2(5)$
   $= 20 + 10$
   $= 30$

5. Do application 1.

6. Introduce exponents as a short way of indicating repeated multiplication with the same factor. Show 4^3 means $4 \times 4 \times 4$. Define: x^n is the product of n factors, each equal to x.

7. Demonstrate:

   a)
   $$\begin{aligned} 2 \times 4^3 + 7 &= 2 \times 4 \times 4 \times 4 + 7 \\ &= 2 \times 64 + 7 \\ &= 128 + 7 \\ &= 135 \end{aligned}$$

   b)
   $$\begin{aligned} (2 \times 4)^3 + 7 &= 8^3 + 7 \\ &= 8 \times 8 \times 8 + 7 \\ &= 512 + 7 \\ &= 519 \end{aligned}$$

8. Expand order of operations to include exponents, as follows:
   - a) Parentheses
   - b) Exponents
   - c) Multiplication and division, in order, from left to right
   - d) Addition and subtraction, in order, from left to right

9. Do application 2.

APPLICATIONS:

1. Evaluate each of the following if $a = 8$, $b = 4$, $d = 3$, $x = 6$, $y = 1$, $z = 5$.

a) $3x + 7$
b) $2dx$
c) $4y - 2$
d) $7y + 4b - d$
e) $3(2b - z)$
f) $\frac{3x - 2}{b}$

2. Evaluate each of the following if a, b, d, x, y and z have the same values as in application 1.

a) d^4
b) $3b^2$
c) $(3b)^2$
d) $2a^2 + a$
e) $4x^2 \times b^3$
f) $\frac{d}{2}(a + b)$

3. a) How can we write 3xxyyz using exponents?
b) Evaluate the above if $x = 2$, $y = 4$, $z = 5$.

SUMMARY:

1. State the complete convention for the order of operations.
2. What is an algebraic expression?
3. Why should we know how to evaluate algebraic expressions?

HOMEWORK:

1. Find the numerical value of the following algebraic expressions if $a = 4$, $b = 8$, $x = 3$, $y = 7$ and $z = 5$.

a) $\frac{11+a}{xz}$
b) $4a - 3x$
c) $2b + y - (z - 1)$
d) $4(b - x) + 2y$
e) $3 + \frac{b}{a} + \frac{3z}{x}$
f) $\frac{2(x+z)+a}{a+b}$
g) b^2
h) $5a^3xy$
i) x^4
j) $3a^2 + b^2$
k) $3x^2$
l) a^3xy^2
m) $3z^2xb$
n) $x^2 + 5x + 4$

2. Express in words:

a) $\frac{a+b+c}{3}$
b) $a^2 + b^2 = c^2$
c) $\frac{2a+b}{3}$

3. Evaluate and tell which is greater:

   a) 2^5 or 5^2?

   b) 7^8 or 8^7?

4. When is the logical sentence: $(r \vee q) \wedge (r \vee s)$

   a) true b) false

Lesson 14 - Ordering Signed Numbers, Absolute Value

AIM: What is the set of signed numbers?

PERFORMANCE OBJECTIVES: The students will be able to...

1. Order and locate signed numbers on a number line.
2. State the opposite of the number.
3. Define what is meant by the absolute value of a number.
4. Find the absolute value of signed numbers.

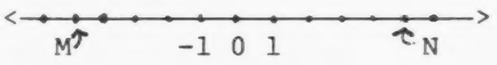
VOCABULARY: positive, negative, absolute value, opposite, is greater than, is less than, integers

MOTIVATION:

At 6PM one evening last winter, the temperature was 10° above zero. During the night, the temperature dropped fifteen degrees. What was the temperature the next morning?

DEVELOPMENT (Include pivotal questions):

1. Have students discuss motivation. Draw a number line. Elicit that 5 units from zero could be to the right or to the left of zero (on a horizontal number line). Points M and N are each 5 units from zero.



2. Review that numbers increase without limit as we move to the right on the number line. A point to the right of zero represents a positive number. A point 5 units to the right of zero (point N above) represents 5 more than 0 and is denoted by 5 or +5 (read, "positive five").

3. How shall we identify numbers to the left of zero? Elicit that they are called negative numbers and are less than zero. Point M represents 5 less than zero and is denoted -5 (read, "negative five").

4. Do applications 1, 2.

5. Elicit that a number line may be extended indefinitely in both directions. What happens to the value of the numbers as we move to the left on a number line? Develop that the numbers represented get smaller as we move to the left on a number line from any position (point) to any other position. Indicate that movement to the left is movement in the negative direction while movement to the right is movement in the positive direction.

6. Review symbols of inequality, > (is greater than), < (is less than).

7. Do applications 3, 4.

8. How many units from zero are -5, +6, +2, $-\frac{2}{3}$, -2? Compare the distances +2 and -2 from zero. Have students note that +2 and -2 are both 2 units from zero, with +2 being 2 units to the right of zero and -2 being 2 units to the left of zero. Develop that such numbers are called opposites.

9. Do application 5.

10. Define the set of whole numbers and their opposites as the set of integers, where
$Z = \{\ldots, -5, -4, -3, -2, -1, 0, +1, +2, +3, +4, +5, \ldots\}$

11. How can we tell which of two opposite numbers is larger? ...which of two numbers is larger? ...which of two numbers is further from zero?
Elicit that of two opposites, the positive one is larger; of two numbers, the one to the right on the number line is larger; of two numbers, the one with the larger value disregarding its sign is further from zero.

12. Develop that we call the distance between the point for a number and zero on a number line the absolute value of the number. Alternately, we may define absolute value as the greater of a number and its opposite.

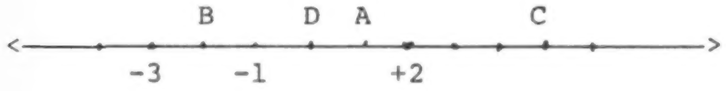
13. Demonstrate that the absolute value of -3 is written $|-3|$, the absolute value of +5 is written $|+5|$.
    $|+5| = 5$ is read, "The absolute value of positive five equals five."
    $|-3| = 3$ is read, "The absolute value of negative three equals three."

14. Do applications 6, 7.

APPLICATIONS:

1. On a number line, what signed number is represented by:
    a) a point 3 units to the right of zero.
    b) a point 8 units to the left of zero.
    c) a point 3 units to the right of 4.
    d) a point 3 units to the right of -5.
    e) a point 4 units to the left of +7.
    f) a point 5 units to the left of +3.

2. Name the points A, B, C, D by a signed number.



3. True or false?
    a) $-20<5$ b) $-10>-3$ c) $0>-9$

4. Use < to order:
    a) -5, +1 b) -6, -8 c) -8, +7

5. What is the opposite of:
    a) -5 b) +8
    c) -2 d) 0

6. True or false?
    a) $|-5| = 5$ b) $|+21| = 21$
    c) $|-11| = -11$ d) $|7| < |-14|$

7. Find the value of (evaluate):
   - a) $|+6| + |-2|$ b) $|-5 + +10|$
   - c) $|-10| - |-5|$

SUMMARY:

1. What is the absolute value of:
   - a) +3 b) +5
   - c) -50 d) 10
2. What is the opposite of:
   - a) +3 b) +5
   - c) -50 d) 10
3. What number is its own opposite?
4. If two numbers are placed on a number line, where will the larger one lie?
5. What integer lies immediately to the right of:
   - a) $5\frac{1}{3}$ b) -4 c) 7 d) $-2\frac{7}{8}$

HOMEWORK:

1. Represent as a signed number:
   - a) a point 8 units to the left of +7
   - b) a point 8 units to the right of +7
   - c) a point 2 units to the right of +2
   - d) a point 5 units to the left of +2
2. Find the value of:
   - a) $|+3| + |-8|$ b) $|-3| + |-1|$
   - c) $|-6| + |+2|$ d) $|-22 + +25|$
   - e) $|+20| - |-6| - |-11|$
3. True or false:
   - a) $-21 = |-21|$ b) $|+13| = 13$
   - c) $|4| + |+2| + |-5| = +1$
4. What number is the opposite of:
   - a) -4 b) 0
   - c) $|+8| + |-2|$

5. Which is between the other two:

   a) -4, +4, -5 b) +43, -11, -22

6. Which positive integers are less than 3?

7. Which negative integers are greater than -5?

8. When is the following logical statement false?

   $(pVq) \longrightarrow r$

Lesson 15 - Addition of Signed Numbers

AIM: How are signed numbers added?

PERFORMANCE OBJECTIVES: The students will be able to...

1. Add signed numbers on a number line.
2. State the rules for adding signed numbers.
3. Add signed numbers using the rules.

VOCABULARY: commutative, addends, absolute value, sum, difference

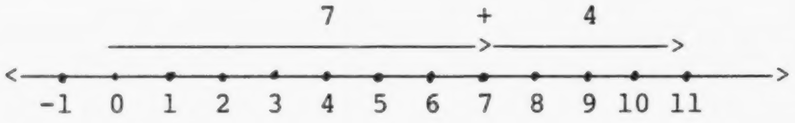
MOTIVATION:

In a football game, what is the net result of each sequence of plays?

1) a gain of 7 yards followed by a gain of 4 yards
2) a gain of 7 yards followed by a loss of 4 yards
3) a gain of 7 yards followed by a loss of 13 yards
4) a gain of 7 yards, a loss of 9 yards, a gain of 5 yards

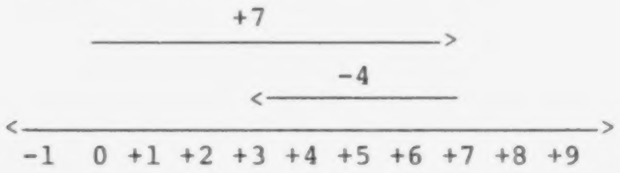
DEVELOPMENT (Include pivotal questions):

1. Elicit that a diagram may be helpful for analysing this situation. How do we add 7 and 4 on a number line?



Review that we start at zero, move 7 spaces to the right, then four more spaces to the right. The answer to the problem is the number which corresponds to the point at which we end (7 + 4 = 11). How could this example have been written using signed numbers? (+7 + +4 = +11)

2. How could we add +7 + -4 on a number line? Elicit:
   a) start at zero and draw an arrow 7 units in a positive direction (to the right)
   b) from that point (+7), move 4 units in a negative direction (to the left)
   c) the point at which we end is the answer to the problem (+3).



3. Similarly, demonstrate solutions to questions 3 and 4 of the motivation.

4. Ask students to find these sums, using the number line:
   a) +3 + -5
   b) -3 + +3
   c) -4 + +7
   d) -4 + -2
   e) -1 + +4 + -5
   f) +7 + -7

   Pivotal Question: What is the relationship between the addends in problems b and f?

   Elicit: The sum of opposites is always zero.

5. By reversing the addends above, show that addition is commutative for the set of integers.

6. Pivotal Question: How would you find the sum of +273 and -491? Students should observe that the number line technique would be impractical for this problem. Suggest that a rule would be convenient and efficient.

7. What pattern do we see in these groups of examples?

   a) +3 + +2 = +5
      +3 + +1 = +4
      +3 + 0 = +3
      +3 + -1 = ?
      +3 + -2 = ?

   b) -3 + -3 = -6
      -3 + -2 = -5
      -3 + -1 = ?
      -3 + 0 = ?
      -3 + +1 = ?

   Elicit that in a), the answers are +2 and +1, respectively. Students should see that the sum is decreasing by 1 in each successive example as is the second addend. In b), the answers are -4, -3 and -2 since the second addend and the sum are each increasing by 1. Use these patterns to extend each list above and below.

8. What is the sign of the sum when adding two signed numbers with the same sign? (The sign of the sum is the same as the sign of the addends.)
9. How can we find the absolute value of the sum? (Review the first two examples in 7a and 7b for models.) (Add the absolute values of the addends.)
10. What rules can we establish for addition of signed numbers with the same sign? Have students generalize: to add 2 signed numbers with the same sign, add their absolute values and keep their common sign.
11. (Use 7a and 7b as models.) If we add two signed numbers with different signs, what is the sign of the sum? (The sign of the sum is the same as the sign of the number with the greater absolute value.)
12. How do we find the absolute value of the sum?
13. What rules can we establish for addition of signed numbers with different signs? Have students generalize: to add two signed numbers with different signs, take the difference of the absolute values of the numbers and use the sign of the number having the greater absolute value.

APPLICATIONS:

1. Compute:

a) +3 + +6　　b) -4 + -37 + +22

c) +3 + +2　　d) -3 + -5 + +89

e) $\begin{array}{r} -8 \\ + \underline{-4} \end{array}$　　f) $\begin{array}{r} -321 \\ + \underline{+490} \end{array}$

2. Compute:

a) -4 + +3　　b) -21 + +53

c) -3 + +1　　d) +27 + -27

e) $\begin{array}{r} -42 \\ + \underline{+37} \end{array}$　　f) $\begin{array}{r} -21 \\ +44 \\ +53 \\ \underline{-94} \end{array}$

SUMMARY:

1. State the rules for adding like and unlike signed numbers.
2. Addition is "commutative". What does this mean?
3. What strategy would you use to add six signed numbers?

HOMEWORK:

1. Compute the sums:

   a) $\begin{array}{r}+5\\ \underline{+3}\end{array}$ b) $\begin{array}{r}+8\\ \underline{-3}\end{array}$ c) $\begin{array}{r}+15\\ \underline{-15}\end{array}$ d) $\begin{array}{r}+23\\ \underline{-42}\end{array}$ e) $\begin{array}{r}+7\\ \underline{0}\end{array}$ f) $\begin{array}{r}0\\ \underline{+8}\end{array}$

2. Compute the sums:

   a) $\begin{array}{r}-8\\ \underline{-9}\end{array}$ b) $\begin{array}{r}-2\\ \underline{+9}\end{array}$ c) $\begin{array}{r}-25\\ \underline{+17}\end{array}$ d) $\begin{array}{r}-19\\ \underline{+19}\end{array}$ e) $\begin{array}{r}-13\\ \underline{0}\end{array}$ f) $\begin{array}{r}0\\ \underline{-4}\end{array}$

3. Compute:

   a) (+5) + (-8) b) (-34) + (+39) c) (-81) + (-45)

   d) -12 + 10 e) (-57) + (+92) f) -216 + +216

4. Add:

   a) -5 + -3 + +12

   b) (-11)+(+4)+(+2)

   c) (-52)+(+19)+(-5)

   d) $\begin{array}{r}-216\\ +23\\ -14\\ \underline{-7}\end{array}$ e) $\begin{array}{r}+84\\ -17\\ -23\\ \underline{+42}\end{array}$ f) $\begin{array}{r}+16\\ +98\\ -112\\ \underline{-57}\end{array}$

*5. If the domain of x is the set of integers, solve each of the following:

   a) $|x| = 7$ b) $|x| = 0$

   c) $|x| = -1$ d) $x + |x| = 6$

   e) $|x+1| = 3$ f) $|x-1| = 3$

Lesson 16 - Subtraction of Signed Numbers

AIM: How are signed numbers subtracted?

PERFORMANCE OBJECTIVES: The students will be able to...

1. Recognize subtraction as addition using the opposite of the subtrahend.
2. State the rule for subtraction of signed numbers.
3. Subtract signed numbers using the rule.

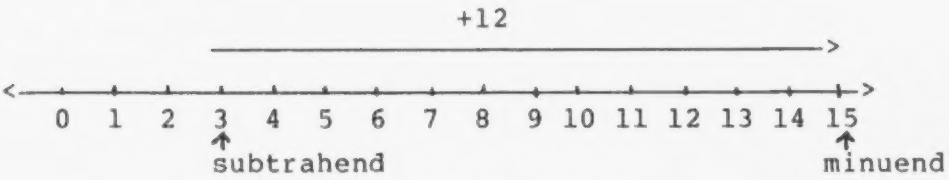
VOCABULARY: minuend, subtrahend, difference

MOTIVATION:

Maria received a $15 prize for a poster she had designed, but had to pay a bill for $3 for the materials she had used.
1) How much money will she have left after paying the bill?
2) What operation did you use to find the answer?

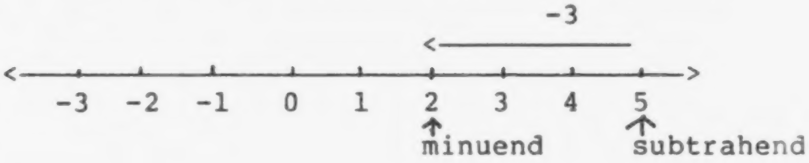
DEVELOPMENT (Include pivotal questions):

1. Discuss motivation. Students should see that the problem may be solved by using either addition (15 + -3 = 12) or subtraction (15 - 3 = 12). In subtraction, the terms of the problem have special names: minuend minus subtrahend equals difference. Stress that subtracting a number and adding its opposite yield the same answer.
2. In subtraction, you seek a number which when added to the subtrahend will produce the minuend. This may be illustrated on the number line by drawing an arrow from the subtrahend to the minuend; the motion represented by the arrow is the difference sought.



(+15) - (+3) = +12

3. For the subtraction example, +2 - +5, what addition example would give the same answer? (+2 + -5) What is the answer? (Students may need to do subtraction of whole numbers on a number line. Show the difference by an arrow from the subtrahend to the minuend.



4. Do application 1.

5. How can we subtract in these examples by converting to an equivalent addition problem?

   a) +7 - +8　　　　b) +3 - +5

   Elicit that we can re-write these examples as

   a) +7 + -8　　　　b) +3 + -5

   These are addition examples which give the same answer as their subtractive counterparts. If we compute the sums, we obtain

   a) +7 + -8 = -1　　　　b) +3 + -5 = -2

6. Do application 2.

7. Elicit the rule for subtraction of signed numbers:

   To subtract signed numbers, change the subtrahend to its opposite and add.

8. Do application 3.

9. How do we check the subtraction example: $\begin{array}{r} +15? \\ -\ \underline{+9} \\ +6 \end{array}$

10. Develop that we check subtraction by adding the subtrahend and the difference. (Addition and subtraction

are inverse operations.) If our answer is correct, the sum is equal to the minuend. Do the subtraction examples in 5a and 5b. Check for the answers we got when we changed the subtrahend to its opposite and added.

$$\text{a)}\quad \begin{array}{r} +7 \\ \underline{-\ \ +8} \\ -1 \end{array} \qquad \text{Check:}\quad \begin{array}{r} +8 \\ \underline{+\ \ -1} \\ +7 \\ \text{(checks)} \end{array} \qquad \text{b)}\quad \begin{array}{r} +3 \\ \underline{-\ \ +5} \\ -2 \end{array} \qquad \text{Check:}\quad \begin{array}{r} +5 \\ \underline{+\ \ -2} \\ +3 \\ \text{(checks)} \end{array}$$

11. Do application 4.

12. Demonstrate procedure for more complicated situations.

$$\text{a)}\quad \begin{aligned} +4 - (+5 - -3) &= +4 - (+5 + +3) \\ &= +4 - +8 \\ &= +4 + -8 \\ &= -4 \end{aligned}$$

$$\text{b)}\quad -(-2 - -3) = -\ (+1) = -1$$

Note: This problem illustrates three different uses of the symbol "-": opposite, negative, subtract.

13. Do applications 5 and 6.

APPLICATIONS:

1. Write as an addition example:
   - a) $+4 - +2$
   - b) $+3 - +3$
   - c) $+1 - +4$
2. Find the answers to the examples in application 1.
3. Subtract as shown:

$$\text{a)}\ \begin{array}{r} +10 \\ \underline{-\ \ +3} \end{array} \qquad \text{b)}\ \begin{array}{r} -8 \\ \underline{-\ \ +4} \end{array} \qquad \text{c)}\ \begin{array}{r} -15 \\ \underline{-\ \ -12} \end{array} \qquad \text{d)}\ \begin{array}{r} -10 \\ \underline{-\ \ -21} \end{array} \qquad \text{e)}\ \begin{array}{r} +11 \\ \underline{-\ \ +15} \end{array} \qquad \text{f)}\ \begin{array}{r} +23 \\ \underline{-\ \ -30} \end{array}$$

4. Check each subtraction in application 3.
5. Compute:
   - a) $17 - 20$
   - b) $50 - (-75)$
   - c) $-5 - (-10)$
   - d) $-(5 - 8)$
6. a) From -12 subtract +12 b) From +42 subtract -33

   c) Subtract +13 from +10 d) Subtract -23 from -10

SUMMARY:

1. What is the rule for addition of signed numbers
   a) with the same sign?
   b) with different signs?
2. What is the rule for subtraction of signed numbers?
3. If the difference is zero, what must be true about the minuend and the subtrahend?

HOMEWORK:

1. What is the opposite of:
   a) +6 b) −4 c) 2 d) 0?
2. Compute the difference:
   a) $\begin{array}{r} +34 \\ \underline{-\ -17} \end{array}$ b) $\begin{array}{r} -12 \\ \underline{-\ -12} \end{array}$ c) $\begin{array}{r} -23 \\ \underline{-\ -21} \end{array}$ d) $\begin{array}{r} -58 \\ \underline{-\ -21} \end{array}$ e) $\begin{array}{r} +32 \\ \underline{-\ +42} \end{array}$
3. From −11 subtract +4.
4. Subtract +4 from −11.
5. Subtract −11 from +4.
6. Evaluate:
   a) $(-13) - (-23)$ b) $42 - (+60)$
   c) $21 - 30 - 4 + 10$

*7. If the domain of each variable is the set of integers, solve:
   a) $|a| < 3$ b) $7 - |b| > 1$
   c) $-7 + |c| = 2$ d) $-7 - |d| = 10$
   e) $-7 - |e| = 10$

*8. If $a<0$ and $b<0$, which is larger: $a + b$ or $a - b$?

Lesson 17 - Multiplication and Division of Signed Numbers

AIM: How are signed numbers multiplied?
How are signed numbers divided?

PERFORMANCE OBJECTIVES: The students will be able to...

1. State the rules for multiplication and division of signed numbers.
2. Compute the product of signed numbers.
3. Find the quotient of two signed numbers.

VOCABULARY: dividend, divisor, factor, product, quotient, like signs, unlike signs, distributive property

MOTIVATION:

George and Fred are following weight control plans. George is gaining 2 pounds per week. Fred is losing 3 pounds per week. Assuming that they meet their weekly goals successfully, use signed numbers to express how their weights at the following times will compare to their current weights:

1) George's weight 4 weeks from now (future)
2) Fred's weight 5 weeks from now

DEVELOPMENT (Include pivotal questions):

1. Discuss motivation. Students should observe that repeated addition of the same number may also be viewed as multiplication. For example, in problem 1, +2 + +2 + +2 + +2 = (+4)(+2). Point out that George could have accomplished the same result in two weeks by gaining 4 pounds per week ((+4)(+2) = (+2)(+4): multiplication is commutative).

Solutions to the motivation may be summarized as:

(+4)(+2) = +8 (Four weeks from now, he will be 8 pounds heavier.)

(+5)(-3) = -15 (Five weeks from now, he will be 15 pounds lighter.)

2. Have students generalize that
   a) the product of two positive numbers is a positive number whose absolute value is equal to the product of the absolute values of the factors.
   b) the product of a positive number and a negative number is a negative number.

3. Have students investigate these multiplication patterns.

   a) +3 x +2 = +6
   +3 x +1 = +3
   +3 x 0 = 0
   +3 x -1 = -3
   +3 x -2 = -6
   +3 x -3 = ?

   b) -3 x +3 = ?
   -3 x +2 = ?
   -3 x +1 = ?
   -3 x 0 = ?
   -3 x -1 = ?
   -3 x -2 = ?

4. From 3a), what rule can we state for multiplication of two signed numbers with different signs? Elicit that the product of two signed numbers with different signs is a negative number whose absolute value is the product of the absolute values of the factors. This verifies the theory arrived at in development 2b.

5. From 3b), what rule can we state for multiplication of two negative numbers? Elicit that the product of two negative numbers is a positive number whose absolute value is the product of the absolute values of the factors. Observe that this is consistent with the use of multiplication in the analysis of the motivation problem. Five weeks ago, Fred was 15 pounds heavier than he is now ((-5)(-3) = +15).

6. As an alternate or ancillary procedure for developing the product of two negative numbers, have students review the distribution of multiplication over addition, the multiplication property of zero and the property of addition of opposites, as follows.

a) Elicit that 7(4+5) = (7x4) + (7x5)

7(9) = 28 + 35

63 = 63

b) Elicit that from 3x0 = 0, 43x0 = 0, -32x0 = 0, it can be concluded that any number times zero equals zero.

c) Review that the sum of a number and its opposite equals zero; for example, -4 + +4 = 0 and +23 + -23 = 0.

d) Since -7 x 0 = 0 then
-7 x (+2 + -2) = 0 because zero was replaced with another numeral for zero.

(-7 x +2) + (-7 x -2) = 0
(-14) + (-7 x -2) = 0

(-7 x -2) must be +14 since it gives a sum of zero when added to -14.

7. Do application 2.

8. How can we multiply several signed numbers?

(+7)(-3)(+4) = ?

Elicit that we follow the order of operations. We multiply +7 by -3 and obtain -21. Then we multiply -21 by +4 and obtain -84.

9. Do application 3.

10. How is twenty-four divided by eight written in symbols? Elicit that division can be written in different ways.

a) 24 ÷ 8 b) $\frac{24}{8}$ c) $8\overline{)24}$

11. Elicit that the answer in a division problem is called the quotient, the number we divide by is called the divisor and the number which is divided is called the dividend.

12. In the division example, $\frac{24}{3} = 8$, how can we check whether 8 is the correct answer? Elicit that we must check division by multiplying the quotient by the divisor. If this product is the dividend, then the example checks.

13. Find the quotient of:
    a) $\frac{+24}{+3}$. Elicit that the quotient is +8 because $(+8)(+3) = +24$.
    b) $\frac{+12}{-4}$. Elicit that the quotient is -3 because $(-3)(-4) = +12$
    c) $\frac{-28}{+7}$. Elicit that the quotient is -4 because $(-4)(+7) = -28$.
    d) $\frac{-22}{-11}$. Elicit that the quotient is +2 because $(+2)(-11) = -22$.
    e) $\frac{7}{0}$. Elicit that there is no answer because there is no number whose product with zero equals 7. Division by zero is not possible.

14. Pivotal Question: How can we summarize the rules we have found for multiplication and division of two signed numbers? Guide the students to develop a compact expression of the rules such as:
    a) The product (or quotient) of two numbers with the same sign will always be positive.
    b) The product (or quotient) of two numbers with different signs will always be negative.
    c) The product of zero with any signed number is zero.
    d) Division by zero is impossible.

15. Do application 4.

APPLICATIONS:

1. Compute:
    a) +11 x +9
    b) +3 x -5
    c) $\begin{array}{r} -4 \\ \underline{\times\ +6} \end{array}$
    d) $\begin{array}{r} +8 \\ \underline{\times\ -7} \end{array}$

2. Compute:
    a) -3 x -8
    b) -7 x -9
    c) $\begin{array}{r} -16 \\ \underline{\times\ -12} \end{array}$
    d) $(-4)^2$
    e) -4^2

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3. Compute each product:

a) (+2)(+4)(+5) b) (+3)(-1)(+2)

c) (-4)(+5)(-10) d) (+4)(+6)(0)(-9)

e) (-3)(-5)(-7)

4. Compute:

a) $\frac{+32}{+4}$ b) -55 ÷ +5 c) 0 ÷ 4

d) $-3\overline{)+18}$ e) $\frac{-49}{-7}$ f) $\frac{-4}{-20}$

SUMMARY:

1. What type of division (multiplication) examples give a positive quotient (product)? Elicit that when dividing (multiplying) two signed numbers with the same sign, the quotient (product) is positive.
2. What type of division (multiplication) examples give a negative quotient (product)? Elicit that when dividing (multiplying) two signed numbers with different signs, the quotient (product) is negative.
3. Is division a commutative operation? Explain.
4. If a product is zero, what may you conclude? Elicit that at least one of the factors must be zero.

HOMEWORK:

1. Compute:

a) +21 x +4 b) -15 x +8

c) $\begin{array}{r} +7 \\ \underline{x\ -9} \end{array}$ d) $\begin{array}{r} -24 \\ \underline{x\ +12} \end{array}$

2. Compute:

a) -6 x -9 b) -38 x -12 c) $\begin{array}{r} -52 \\ \underline{x\ -26} \end{array}$

d) $(-5)^2$ e) $(+4 + -18)^2$

3. Compute:

a) (+3)(-7)(+8) b) (-8)(-9)(+6)

c) (-5)(-12)(-22) d) (+2)(-3)(+4)

4. Compute:

a) $-72 \div -12$

b) $+132 \div -11$

c) $\frac{+52}{+13}$

d) $+7\overline{)-84}$

*5. If the domain of each variable is the set of integers, solve:

a) $-2a = 6$

b) $-2b = -6$

c) $2c = -6$

d) $\frac{6}{d} = -2$

e) $\frac{-6}{e} = -2$

f) $\frac{-6}{f} = 2$

g) $\frac{-6}{-g} = 2$

h) $\frac{-6}{-h} = -2$

i) $\frac{6}{3- x} = 3$

Lesson 18 - Evaluation of Algebraic Expressions Containing Signed Numbers

AIM: How do we evaluate algebraic expressions containing signed numbers?

PERFORMANCE OBJECTIVES: The students will be able to...

1. Recall the procedure for evaluating algebraic expressions.
2. Use the rules for operations with signed numbers and the rules for order of operations to evaluate algebraic expressions.
3. Apply the Commutative, Associative and Distributive Principles.

VOCABULARY: commutative, associative, distributive properties, reciprocal, multiplicative inverse, simplify

MOTIVATION:

1. If $x = 3$ and $y = 4$, evaluate: $3x^2 - 2(y + 9)$.
2. If $x = 4$ and $y = -2$, which is larger: $x^2 + 3y$ or $x^2 - 3y$?

DEVELOPMENT (Include pivotal questions):

1. Go over motivation question 1. Review the procedure for evaluating algebraic expressions (Lesson 13). How do we evaluate algebraic expressions? Elicit that we:
   a) replace the variable with its given value at each occurrence;
   b) follow the order of operations.

   For $3x^2 - 2(y + 9)$ if $x = 3$ and $y = 4$,

$$\begin{aligned} 3(3^2) - 2(4 + 9) &= 3(9) - 2(13) \\ &= 27 - 26 \\ &= 1 \end{aligned}$$

For question 2,

$$\begin{aligned} x^2 + 3y &= (+4)^2 + 3(-2) \\ &= +16 + -6 \\ &= +10 \end{aligned} \qquad \begin{aligned} x^2 - 3y &= (+4)^2 - 3(-2) \\ &= +16 - -6 \\ &= +16 + +6 \\ &= +22 \end{aligned}$$

Pivotal questions: Will the expression on the right always have a greater value? What value of y would make the expressions equal? What values of y would make the expression on the left greater?

2. What is the value of $3a^2 - (ab)^2$ if $a = -3$ and $b = +2$? Develop that we follow the procedure as shown above.

$$\begin{aligned} 3a^2 - (ab)^2 &= 3(-3)^2 - (-3 \times +2)^2 \\ &= 3(+9) - (-6)^2 \\ &= +27 - +36 \\ &= -9 \end{aligned}$$

3. Do application 1.

4. Review commutative, associative, distributive properties, multiplication properties of one and zero.
   What property is illustrated by each of the following examples?
   - a) $6 + 4 = 4 + 6$
   - b) $(7 + 3) + 9 = 7 + (3 + 9)$
   - c) $9 \times 7 = 7 \times 9$
   - d) $(3 \times 2) \times 5 = 3 \times (2 \times 5)$
   - e) $3(4 + 6) = (3 \times 4) + (3 \times 6)$
   - f) $34 \times 1 = 34$
   - g) $23 \times 0 = 0$

5. How can these properties be used to rewrite the following expressions?
   - a) $a + b$
   - b) $r + (s + t)$
   - c) $x(3 + y)$
   - d) $a(bc)$
   - e) $4(-3x)$
   - f) $1(y)$
   - g) $-(2x - y)$
   - h) $(+9x) + (-15x)$

6. Do application 2.

7. Review definition of reciprocals (multiplicative inverses) and the property of multiplication of reciprocals.

a) Have students multiply $\frac{1}{4} \times 4$; $2\frac{3}{5} \times \frac{5}{13}$.

b) Have the students note that the product in each case is 1.

c) Define reciprocals as two numbers whose product is one.

Emphasize that each number is the reciprocal of the other. In the above, $\frac{1}{4}$ is the reciprocal of 4 and 4 is the reciprocal of $\frac{1}{4}$. Similarly, $2\frac{3}{5}$ and $\frac{5}{13}$ are reciprocals of each other. What is the reciprocal of -7? $-2\frac{1}{3}$?

8. What property justifies each of the equalities?

$$\begin{aligned} 11 \times (3 \times \tfrac{1}{11}) &= 11 \times (\tfrac{1}{11} \times 3) \quad \text{(commutative property)} \\ &= (11 \times \tfrac{1}{11}) \times 3 \quad \text{(associative property)} \\ &= (1) \times 3 \quad \text{(product of reciprocals)} \\ &= 3 \quad \text{(multiplication property of one)} \end{aligned}$$

9. Do application 3.

10. How can 4y + 2y be combined? Demonstrate that we can write:

$$\begin{aligned} 4y + 2y &= (4 + 2)y \\ &= (6)y \\ &= 6y \end{aligned}$$

Discuss that replacing 4y + 2y with the equivalent form 6y is called a simplification of 4y + 2y.

APPLICATIONS:

1. Evaluate if $a = -2$, $b = +4$, $d = -5$, $x = -4$, $y = +2$ and $z = -3$.

a) $4a$	b) $-3b$
c) xd	d) $2ab$
e) b^2	f) $2x^2y^3$
g) $z - x^2$	h) $(a + b)(x + y)$
i) $2(3a + 2b + d) - 2z$	j) $x - z^3$

2. Simplify by using the commutative, associative and distributive properties. (Distribution factor.)

a) $3(x + 5)$	b) $2 + (6 + x)$

c) $-2(4 - 2x)$ d) $14a + 7b$
e) $2a - 6y$ f) $3x + 2x$

3. Use the associative, commutative and distributive properties to simplify the following.
   a) $3(x + 2) + 5$ b) $(4a)(2a)$
   c) $-4(63 x -\frac{1}{4})$

SUMMARY:

1. Which is larger: the opposite of 8 or the reciprocal of 8?
2. Which operations are commutative? Explain.
3. Is subtraction associative? Explain.
4. Does addition distribute over multiplication? Explain.
5. Specify two numbers, each of which is equal to its reciprocal.
6. What number has no reciprocal?
7. Summarize the procedure for evaluating algebraic expressions.

HOMEWORK:

1. Evaluate if $a = 4$, $b = 2$, $c = -3$, $d = -2$, $x = -5$ and $y = 3$:
   a) $5(a + b)$ b) $-5y - c^3$
   c) dx d) $7ca - d^2$
   e) $(2y)^2$ f) $4x - 3d$
   g) $3a^2y^2$ h) $(2a - b)(c + d)$
2. Simplify:
   a) $5y + (4y + 4)$ b) $5(b + 2) - 10$
   c) $6(a + 2b) + 2(2a + b)$
3. Using the distributive property, write each of the following without parentheses.
   a) $4(x + z)$ b) $-3(a + b)$
   c) $(3 + 4y)x$

4. Solve:
   - a) $2(x + 1) = 12$ b) $-2(x + 1) = 12$
   - c) $2(x + 1) = -12$

5. Is $2|x| = |2x|$ for all numbers x?

6. Is $-2|x| = |-2x|$ for all numbers x?
   Is it true for any value of x?

7. Give an example which illustrates that division is not associative.

Lesson 19 - Addition and Subtraction of Monomials

AIM: How do we add and subtract monomials?

PERFORMANCE OBJECTIVES: The students will be able to...

1. Identify a monomial.
2. Identify:
   a) coefficients
   b) variables
   c) exponents
3. State the procedure for addition and subtraction of monomials.
4. Add and subtract like monomials.

VOCABULARY: monomial, literal factor, like monomials, unlike monomials, coefficients, base, exponent, polynomial, terms of an algebraic expression

MOTIVATION: Sharon paid 3x dollars for a computer and 2x dollars for software and accessories. Write the simplest possible expression for the total cost.

DEVELOPMENT (Include pivotal questions):

1. Go over motivation. $3x + 2x = x(3 + 2)$
   $= x(5)$
   $= 5x$
2. Develop that 3x + 2x is an algebraic expression in which 2 parts are added. Similarly, 9x - 2y is an algebraic expression in which one part is subtracted from the other. These parts are called terms.
3. How many terms are in the expression, 9x + 4y - 3z + 4? Elicit that there are four terms in the expression and

that they are separated from each other by + or − (addition or subtraction signs). Identify the terms as 9x, 4y, 3z and 4.

4. Define a monomial as an algebraic expression containing a single (one) term. (Alternatively, a monomial is a numeral or a variable or the indicated product or quotient of a numeral and one or more variables.)

5. What are the parts of a monomial? The monomial, $8x^2$ means 8 times x^2. 8 is called the coefficient or numerical coefficient, x is the variable and 2 is the exponent. The part of the term which is multiplied by the numerical coefficient is called the literal factor. In $8x^2$, x^2 is the literal factor.

6. If a term contains an exponent, then the part of the term for which the exponent applies is called the base of the exponent. In $8x^2$, x is the base and its exponent is 2.

7. Do applications 1, 2, 3.

8. How are the terms 3x and 3y similar? (They have the same coefficients.) How are they different? (They have different literal factors.)

9. How are the terms 5xy and 34xy similar? (They have the same literal factors.) How are they different? (They have different coefficients.)

10. Define like terms as terms which have the same literal factors.

11. Terms with different literal factors are called unlike terms.

12. Do application 4.

13. How can the like terms in the expression 3x + 4x be combined?

    Demonstrate that:

    a) $$\begin{aligned} 3x + 4x &= (3 + 4)x \\ &= (7)x \\ &= 7x \end{aligned}$$

b) $$\begin{aligned} 8m - 8m &= (8 - 8)m \\ &= (0)m \\ &= 0 \end{aligned}$$

Demonstrate that this is the equivalent of $8m + -8m$. Since this equals zero, then $8m$ and $-8m$ are opposites. Generalize that like terms whose coefficients are opposites are also opposites. (Their sum is zero.)

c) $$\begin{aligned} 7y^2 - 2y^2 &= (7 - 2)y^2 \\ &= (5)y^2 \\ &= 5y^2 \end{aligned} \quad \text{or} \quad \begin{array}{r} 7y^2 \\ \underline{-2y^2} \\ 5y^2 \end{array}$$

14. Develop the rules for addition and subtraction of like monomials:

    To add like monomials, add their numerical coefficients and multiply this sum by their common literal factor. To subtract like monomials, change the sign of the subtrahend to its opposite and add.

15. Do applications 5, 6.

APPLICATIONS:

1. Name the numerical coefficient of each of the following terms:
   a) $8x$　　b) $5y$
   c) $\frac{1}{2}x$　　d) $-3z$
   e) b

2. Name the base of each of the following terms:
   a) $3x^5$　　b) x^4
   c) $2y^3$　　d) 4^2

3. Name a) the numerical coefficient and
   b) the literal factor and
   c) the variables of the following terms:
   a) $5s$　　b) $7y^2$
   c) $3xy^2$　　d) $3x^2$
   e) $-8x^2y^3$　　f) $9y^2$
   g) $-8xy^2$　　h) $3x^2y$

4. Which of the terms in application 3 are like terms?

5. Combine like terms:

a) $16x^2 - 2x^2$ b) $2a + 7a$

c) $-5xy + -3xy$ d) $\begin{array}{r} +9x \\ \underline{-9x} \end{array}$

6. Compute the difference:

a) $\begin{array}{r} +8y \\ \underline{-\ +3y} \end{array}$ b) $\begin{array}{r} -5x^2 \\ \underline{-\ -3x^2} \end{array}$

c) $\begin{array}{r} 14t \\ \underline{-\ \ -t} \end{array}$ d) $\begin{array}{r} +4xy \\ \underline{-\ +4xy} \end{array}$

e) $\begin{array}{r} b \\ \underline{-\ -3b} \end{array}$ f) $\begin{array}{r} 2x \\ \underline{-\ +5x} \end{array}$

SUMMARY:

1. What is a term?

2. What are like terms?

3. What are opposites?

4. State the rule for combining like terms.

HOMEWORK:

1. Which of the following are monomials?

a) $4x - 2w$ b) $3x^2$

c) $-5xy$ d) $31x^2y^3z$

e) $3x^2 - 2y$

2. Name the numerical coefficients of the following monomials:

a) $4x$ b) $-10z$

c) b d) $7x^2y$

e) abc f) $-x$

3. Name the variables:

a) $4x$ b) $-10z$

c) b d) $7x^2y$

e) abc f) $-x$

4. What is the exponent in each of the following monomials?

a) x^5 b) $-21y^2$

c) $3x^4$ d) $3x$

e) $-4x^3$

5. Add:

a) $\begin{array}{r}+12x^2\\ \underline{+\ 5x^2}\end{array}$ b) $\begin{array}{r}+32y\\ \underline{-18y}\end{array}$ c) $\begin{array}{r}+22bc\\ \underline{-52bc}\end{array}$

d) $\begin{array}{r}-43xy^4\\ \underline{-13xy^4}\end{array}$ e) $\begin{array}{r}-8r\\ \underline{+9r}\end{array}$ f) $\begin{array}{r}-12mn\\ \underline{+12mn}\end{array}$

6. Subtract:

a) $\begin{array}{r}+4y\\ \underline{+y}\end{array}$ b) $\begin{array}{r}-5st\\ \underline{-2st}\end{array}$

c) $\begin{array}{r}0\\ \underline{+3x}\end{array}$ d) $\begin{array}{r}2x^2\\ \underline{x^2}\end{array}$

e) $\begin{array}{r}+xy\\ \underline{+3xy}\end{array}$ f) $\begin{array}{r}-4x^2y^3\\ \underline{-4x^2y^3}\end{array}$

*7. Solve:

a) $3(2x + 3) - 6x = x$

b) $4(5 - y) + 4y = y$

8. The dimensions of a rectangular garden are $3x$ and $8x$. If you surround the garden with a fence, how long will the fence be?

Lesson 20 - Multiplication of Monomials

AIM: How do we multiply monomials?

PERFORMANCE OBJECTIVES: The students will be able to...

1. State the rules for multiplying monomials with regard to coefficients, variables and exponents.
2. Multiply monomials containing varied types of coefficients and positive integral exponents.

VOCABULARY: exponent, coefficient, base, factor, monomial

MOTIVATION:

1. Write an equivalent expression without exponents.
   a) $12x^5$ b) $10x^2y$
2. Which is larger: $(2^3)(2^4)$ or 2^{12}?

DEVELOPMENT (Include pivotal questions):

1. Go over motivation. Why are exponents used? Review that the use of exponents is a short way to show multiplication using the same factor.
2. Review that in examples such as x times y = xy, the fact that there is no operational symbol between the x and y indicates x is multiplied by y.
3. How can we use the commutative and associative properties to simplify (5c)(4c)? Develop that
   $$\begin{aligned}(5c)(4c) &= (5)(4)(c)(c)\\ &= 20c^2\end{aligned}$$
4. Similarly, develop that
   $$\begin{aligned}(2x)(4x)(6xy) &= (2)(4)(6)(x)(x)(x)(y)\\ &= 48x^3y\end{aligned}$$

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5. Write $(x^3)(x^2)$ without exponents. Show that $(x^3)(x^2) = (x)(x)(x) \cdot (x)(x) = x^5$. Generalize: $(x^a)(x^b) = x^{a+b}$.

6. Do application 1.

7. Have the students note that in application 1a and 1b the base is the same as in 5 above. How do we multiply powers of the same base? Elicit that to multiply two powers of the same base, the product has the same base and the exponent is the sum of the exponents of the factors. Emphasize that the rule applies only when the base is the same:

   x^2y^3 cannot be simplified because the bases of x^2 and y^3 are different.

8. What is the product of $(-6x^2y)$ and $(+3xy)$? Develop that

$$\begin{aligned}(-6)(x)(x)(y)(+3)(x)(y) &= (-6)(+3)(x^2)(x)(y)(y)\\ &= -18x^3y^2\end{aligned}$$

   Elicit that to multiply monomials,
   - a) multiply numerical coefficients using the rules for multiplication of signed numbers;
   - b) add exponents where the base is the same.

9. Do application 2.

APPLICATIONS:

1. Compute the product of the following monomials:

   a) $(2a)(5a)$　　b) $(4x)(5x)(3x)$
   c) $(3xy)(4y)(5xy)$　　d) b^2b^4

2. Compute:

   a) $(6)(-2a)$　　b) $(-4r)(5rs)^2$
   c) $(-2b^3)(-3b^5)$　　d) $(+7ab)(-2c)$
   e) $(+4x^2y^3)(7x^3y^4)$　　f) $(+5a^2b^3)(3b)$
   g) $(-3xy^2)(-4x^2y^3)$　　h) $(-3)(+4x)(-3x)$
   i) $(-2x)^2$　　j) $(+5x^4)(-2x)(-b^3)$

SUMMARY:

1. How many terms are in the product of monomials? Have students conclude that when any number of monomials are

multiplied, the product is a monomial.

2. What is the value of $(2x^5)^3$?

3. What is the quotient of x^8 divided by x^2?

HOMEWORK:

1. Compute:
   - a) $(+5)(-2x)$
   - b) $(+4x)(-3y)$
   - c) $(-3r)(-s)$
   - d) y^6y^4
   - e) a^2a
   - f) t^5t^3t
   - g) $(4y)(2y^4)$
   - h) $(-5x^2)(3xy)$
   - i) $(-2r^2t^2)(-5r^2t^3)$
   - j) $(+4z)(-3z^4)(-4z^3)$

2. Compute:
   - a) $(-4x)(-5y)$
   - b) $(+6r)(-3st)$
   - c) $(3ab)(6cd)(-2e)$
   - d) $(20x^3)(-4x^6)$
   - e) $(-8a^4b)^3$
   - f) $(5x^5y^4)(-6x^4y^3)(4x^2)$
   - g) $(+4x^2)^3$
   - h) $(-3y)^2$

*3. Solve:
   - a) $|2x + 3| = 0$
   - b) $|2x + 3| = 1$

Lesson 21 - Division of Monomials

AIM: How do we divide monomials?

PERFORMANCE OBJECTIVES: The students will be able to...

1. State the rules for division of monomials with regard to coefficients, variables and exponents.
2. Divide monomials containing varied types of coefficients and integral exponents leading to a positive exponent in the quotient.

VOCABULARY: divisor, dividend, quotient, power, exponent, coefficient, literal factor

MOTIVATION:

1. Which fraction has the least value?

   a) $\frac{+6}{-3}$ b) $\frac{-3}{-3}$

   c) $\frac{x}{x}$ $(x \neq 0)$ d) $\frac{(3)(4)}{(7)(4)}$

2. Tom and Anna do some work in the neighborhood, earning together $6x^2$ dollars. If they share this money equally, how much should each of them receive?

DEVELOPMENT (Include pivotal questions):

1. Go over motivation to review division of signed numbers and the meaning of exponents. In question 1, d has the least absolute value, but a is smaller, since it is less than zero. Students should observe that any nonzero number divided by itself equals positive one. A number line may be used to clarify the solution.

2. How do we check answers in division problems? Elicit that we check division by multiplying the quotient by the divisor. If we obtain the dividend, the example checks. Check question 2: $(2)(3x^2) = 6x^2$

3. Divide:

   $$\frac{x^5}{x^2} = \frac{x \cdot x \cdot x \cdot x \cdot x}{x \cdot x}$$

   Students will cancel common factors and obtain x^3. Ask, "How do we check if $\frac{x^5}{x^2} = x^3$?" Elicit that $(x^3)(x^2) = x^5$. Follow a similar procedure to show that $\frac{y^9}{y^3} = y^6$. In the first case, what is the arithmetic relationship of 5, 2 and 3? What is the arithmetic relationship of 9, 3 and 6? Elicit that when dividing powers with the same base, keep the base and subtract the exponent of the divisor from the exponent of the dividend to get the exponent of the quotient.

4. Do application 1.

5. How do we divide monomials containing coefficients which are signed numbers and having different literal factors? Consider $\frac{-24x^4y^2}{+6x^2y}$

   After students suggest some answers, ask, "How can we check to see if the answer is correct?" Elicit that the product of the quotient and divisor must equal the dividend.

6. Develop that the rule for dividing monomials is to
   a) divide the numerical coefficients to obtain the numerical coefficient of the quotient.
   b) divide powers of the same base by subtracting the exponents to find the literal factor.
   c) multiply the numerical coefficient by the literal factor.

7. Do application 2.

8. If appropriate, explore an example such as $\frac{x^5}{x^5}$. Students will suggest that a number divided by itself equals 1,

except that the divisor cannot be zero. In addition, by applying the rules just developed, we obtain x^0. Discuss that $x^0 = 1$ for all values of x by definition, except that most mathematicians agree that 0^0 has no meaning.

9. What does x^1 mean? Discuss that x^1 means that x is a factor once. This is the same as x. Therefore $x^1 = x$.

APPLICATIONS:

1. Compute:
   - a) $\frac{x^9}{x^4}$
   - b) $\frac{r^{21}}{r^{14}}$
   - c) $\frac{y^8}{y}$
   - d) $\frac{z^3}{z^3}$

2. Compute:
   - a) $\frac{-18x}{3}$
   - b) $\frac{12a^4}{6a^2}$
   - c) $\frac{24y^3z^8}{-6y^2}$
   - d) $(-42xyz^2) \div (-7xyz)$
   - e) $(24x^5y^2) \div (3x^4y)$

SUMMARY:

1. When dividing monomials, what are the rules for finding
   - a) the coefficient of the quotient and
   - b) the literal factor of the quotient?
2. Give an example of a pair of monomials whose quotient is two.

HOMEWORK:

1. Compute:
   - a) $c^7 \div c^3$
   - b) $21x^5y^2 \div -3$
   - c) $\frac{36x^4z^7}{9x^3z^2}$
   - d) $\frac{-48a^4b^2c^5}{-16a^4b^2c^5}$

2. Compute and check:

a) $35x^4 \div 5x$

b) $\dfrac{-32a^5b^3}{8a^2b^3}$

c) $\dfrac{-63y^{11}z^9}{-9y^4z^6}$

d) $\dfrac{42rs^5}{-7s^4}$

3. If a = 2, b = 3 and c = 5, compute:

a) $\dfrac{4a+b-c}{3}$

b) $\dfrac{a(b+c)}{-4}$

c) $\dfrac{-3a^2b}{-ab}$

d) $-\dfrac{a-2b}{b-2c}$

*4. Compute:

a) $\dfrac{x^3y^2}{xy^5}$

b) $\dfrac{8x^3y^2}{4x^3y^4}$

c) $\dfrac{3x^5y}{6xy^2}$

Lesson 22 - Negative/Zero Exponents

AIM: What is the meaning of a negative exponent?
What is the meaning of a zero exponent?

PERFORMANCE OBJECTIVES: The students will be able to...

1. Recall the rule for division of monomials.
2. Explain the meaning of a negative exponent.
3. Explain the meaning of a zero exponent.
4. Rewrite monomials containing negative exponents as monomials with positive exponents.

VOCABULARY: exponent, base, power

MOTIVATION:

Suppose that the area of a rectangle is known to be 2^5. Find the width when the length is:

a) 2^3 b) 2^5 c) 2^7

DEVELOPMENT (Include pivotal questions):

1. Discuss motivation. Two different strategies can be used to solve this problem; the class should review both methods. Using direct evaluation, the answers 4, 1, $\frac{1}{4}$ may be found. Using the rule for division of powers, the same answers are expressed as 2^2, 2^0, 2^{-2}. Students should realize that $2^{-2} = \frac{1}{2^2}$.
2. Recall the rule for dividing powers with like bases:
   $x^a \div x^b = x^{a-b} \quad (x \neq 0)$
   Suppose a = b; then
   $x^a \div x^b = x^a \div x^a = x^{a-a} = x^0$
   But $x^a \div x^a$ is a division of a number by itself. What is

the result when you divide a number ($\neq 0$) by itself? For example, $\frac{3}{3}$ is equal to one. So, $x^a \div x^a = 1$.
Therefore, we have the following definition: $x^0 = 1, x \neq 0$.
Since $x^0 = 1$ is based upon division and division by zero is impossible, 0^0 is not defined.
Using the definition $x^0 = 1$, ($x \neq 0$) permits us to observe that

$$\begin{aligned} 7^0 &= 1 \\ (-7)^0 &= 1 \\ (7x)^0 &= 1 \\ (-7x)^0 &= 1 \end{aligned}$$

However, $7x^0 = 7$, because $7x^0 = 7(x^0) = 7(1) = 7$.

3. Do application 1.

4. By the division rule, $\frac{x^5}{x^7} = x^{5-7} = x^{-2}$.
   But $\frac{x^5}{x^7} = \frac{\cancel{x}\cdot\cancel{x}\cdot\cancel{x}\cdot\cancel{x}\cdot\cancel{x}}{\cancel{x}\cdot\cancel{x}\cdot\cancel{x}\cdot\cancel{x}\cdot\cancel{x}\cdot x\cdot x} = \frac{1}{x\cdot x} = \frac{1}{x^2}$
   Therefore $x^{-2} = \frac{1}{x^2}$.
   We have the following definition:

   $$x^{-a} = \frac{1}{x^a} \quad (x \neq 0) \qquad (x^{-a} \text{ is the reciprocal of } x^a.)$$

   Using the above definition and remembering the rule for multiplying powers of the same base ($x^a \cdot x^b = x^{a+b}$), observe that $3^{-5} = \frac{1}{3^5}$ and

   $$3^2 \cdot 3^{-5} = 3^2 \cdot \frac{1}{3^5} = \frac{3^2}{3^5} = \frac{\cancel{3}\cdot\cancel{3}}{\cancel{3}\cdot\cancel{3}\cdot 3\cdot 3\cdot 3} = \frac{1}{3\cdot 3\cdot 3} = \frac{1}{3^3} = 3^{-3}$$

5. Do application 2.

6. Do applications 3 and 4.

APPLICATIONS:

1. Find the value of each of the following expressions.
   a) b^0, ($b \neq 0$)
   b) -5^0
   c) 2^0
   d) $(.2)^0$
   e) $(2^5)^0$
   f) $2x^0$, ($x \neq 0$)
   g) $(2x)^0$, ($x \neq 0$)
   h) $(\frac{1}{8})^0$
   i) $(\frac{7}{9})^0$

2. Compute each of the following.
    - a) 3^{-1}
    - b) 4^{-2}
    - c) 2^{-3}
    - d) 8^{-2}
    - e) 2^{-1}
    - f) y^{-2} if $y = \frac{3}{4}$
3. Write an equivalent expression using only positive exponents.
    - a) a^{-9}
    - b) x^{-3}
    - c) b^{-1}
    - d) y^{-4}
4. Using laws of exponents, evaluate or simplify:
    - a) $x^0 \cdot x^5$
    - b) $4^3 \cdot 4^{-2}$
    - c) $2^{-3} \cdot 2^{-4}$
    - d) $4^3 \div 4^0$
    - e) $7^{-3} \div 7^{-5}$
    - f) $(x^{-3})^2$

SUMMARY:

1. If $y \neq 0$, find the value of y^0.
   What is the value of any number or variable (base $\neq 0$) having an exponent of zero?
2. Express y^{-2} without using a negative exponent.
   How can we change a variable with a negative exponent to an equivalent expression with a positive exponent?
3. What is the product of y^n and y^{-n}?

HOMEWORK:

1. Write each of the following numbers without using any exponents.
    - a) 2^0
    - b) 6^{-2}
    - c) 8^0
    - d) 5^3
    - e) 7^{-1}
    - f) -3^0
    - g) -9^{-1}
    - h) -3000^{-1}
    - i) 16^0
    - j) -1^4
2. Write each of the following expressions using only positive exponents.
    - a) x^{-8}
    - b) $\frac{1}{x^{-6}}$
    - c) $(3xy)^{-2}$
    - d) x^{-1}
    - e) $(3x^{-3})^2$
    - f) $(\frac{x}{y^3})^{-2}$

3. Using Laws of Exponents evaluate:

a) $6^5 \div 6^2$ b) $2^4 \cdot 2^6$ c) $3^{-4} \div 3^5$

4. Write an equivalent expression that has no denominator.

a) $\frac{1}{a^2}$ b) $\frac{bh}{2}$

c) $\frac{h(b+b')}{2}$ d) $\frac{z}{2x^2y}$

5. Evaluate each expression when $x = 2$, $y = 3$, $z = -2$.

a) $\frac{6x^2y^5}{2xy^3}$ b) $\frac{4x^2yz^3}{4x^2yz^3}$ c) $\frac{2xy^3}{6x^2y^5}$

Lesson 23 - Addition/Subtraction Polynomials

AIM: How can we add and subtract polynomials?

PERFORMANCE OBJECTIVES: The students will be able to...

1. Define and identify a polynomial.
2. Recognize like terms from two or more polynomials.
3. Recall the rules for addition and subtraction of monomials.
4. Apply rules for addition and subtraction of monomials to addition and subtraction of polynomials.

VOCABULARY: polynomial, binomial, trinomial, degree of a polynomial, subtrahend, minuend, difference, descending order, ascending order, like terms, additive inverse

MOTIVATION:

1. Simplify $3x^2 + 2x + 7x^2 + 5x$
2. What polynomial must be added to $5x^2 - 3x + 7$ to produce zero?

DEVELOPMENT (Include pivotal questions):

1. A monomial, such as $3y^2$, has one term (mono = one). A polynomial is a sum of monomials (poly = many). A polynomial of two unlike terms, such as $3a - 7b$, is called a binomial (bi = two). A polynomial of three unlike terms, such as $3x^2 + 2x - 5$, is called a trinomial (tri = three). The degree of a polynomial is the greatest of the degrees of its terms. For example, $x^3 - 3x^2 + 2x + 5$ has a degree of 3.

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2. A polynomial is written in standard form when its terms are arranged in descending order. Descending order is present when the exponents of a particular variable decrease as we move from left to right. Ascending order would be the reverse.

   Find the sum of 731, 42 and 123. Observe that each number may be written as a polynomial.

$$\begin{array}{rcrcr} 7(100) & + & 3(10) & + & 1 \\ & & 4(10) & + & 2 \\ 1(100) & + & 2(10) & + & 3 \\ \hline 8(100) & + & 9(10) & + & 6 \end{array}$$

   Elicit that computing the sum of two or more polynomials involves computing the sum of their like terms.

3. Ask students to add $7x^2 + 3x + 1$, $4x + 2$ and $x^2 + 2x + 3$. To add polynomials, we may write them vertically, arranging them in descending order, and then add the like terms. Each column is done separately.

   Example:

$$\begin{array}{rcrcr} 7x^2 & + & 3x & + & 1 \\ & & 4x & + & 2 \\ x^2 & + & 2x & + & 3 \\ \hline 8x^2 & + & 9x & + & 6 \end{array}$$

   Remind students that a sign not written is understood to be positive.

4. To add polynomials horizontally, we add like terms. For example, ask students to add

   $3x^2 + 7x - 1$ and $2x^2 - 28x -11$.

   Vertically:

$$\begin{array}{rcrcr} 3x^2 & + & 7x & - & 1 \\ 2x^2 & - & 28x & - & 11 \\ \hline 5x^2 & - & 21x & - & 12 \end{array}$$

   Horizontally:

$$3x^2 + 7x - 1 + 2x^2 - 28x -11 = 5x^2 - 21x -12$$

5. Have students leave space for missing terms. For example, to add $3a^2 + 5a - 7$, $8a^2 - 3$, $7 - 4a$, how do we proceed?

$$\begin{array}{r}
3a^2 + 5a - 7 \\
8a^2 - 3 \\
\underline{ - 4a + 7} \\
11a^2 + a - 3
\end{array}$$

1) arrange in descending powers of a
2) arrange like terms in same column
3) add like terms in each column

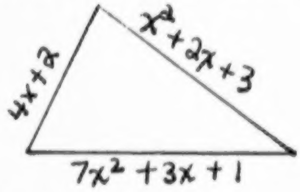
How do we check the answer?

1) Addition can be checked by adding in the opposite direction.
2) Addition can be checked by assigning a numerical value to the variable and evaluating each expression. For example, let a = 2:

$$\begin{array}{lcl}
3a^2 + 5a - 7 & : & 3(2)^2 + 5(2) - 7 = 3(4) + 10 - 7 = 15 \\
8a^2 + 0 - 3 & : & 8(2)^2 + 0 - 3 = 8(4) - 3 = 29 \\
\underline{0 - 4a + 7} & \underline{:} & \underline{0 - 4(2) + 7 = - 8 + 7 = -1} \\
11a^2 + a - 3 & : & 11(2)^2 + 1(2) - 3 = 11(4) + 2 - 3 = 43
\end{array}$$

Discuss the disadvantage of selecting 0 or 1 as numbers for checking. The first method is superior because it is more general and less time consuming.

6. Find the perimeter of the triangle shown and simplify.



7. Do applications 1, 2 and 3.

8. To subtract polynomials, we use a procedure similar to that in subtracting monomials. We add the opposite (additive inverse) of the subtrahend to the minuend. That is, we change the operation subtraction sign to an operation addition sign and the signs of the subtrahend polynomial to their opposites.

$$\begin{array}{r}
4x^2 + 2x + 3 \\
\underline{\overset{+}{\ominus}\overset{-}{\oplus}3x^2 \overset{-}{\oplus} 5x \overset{+}{\ominus} 1} \\
x^2 - 3x + 4
\end{array}$$

How can we check subtraction? (by adding the subtrahend to the minuend.)

Check:

$$\begin{array}{r} 3x^2 + 5x - 1 \\ \underline{x^2 - 3x + 4} \\ 4x^2 + 2x + 3 \end{array}$$

Remind students of the various ways of stating subtraction problems. They can be phrased differently. Solve the following:

a) From $4x^2 + 2x + 3$, subtract $3x^2 + 5x - 1$

b) Subtract $3x^2 + 5x - 1$ from $4x^2 + 2x + 3$.

c) How much greater than $3x^2 + 5x - 1$ is $4x^2 + 2x + 3$?

d) Simplify $4x^2 + 2x + 3 - (3x^2 + 5x - 1)$.

9. Do application 4.

APPLICATIONS:

1. In exercises a through c, state the degree of the polynomial.

   a) $x^2 + 2$ b) $x^3 - 4x + 1$ c) $5x + 9$

   In d through f, state whether the expression is a monomial, binomial or trinomial.

   d) $8y$ e) $6x^2 + 2x - 1$ f) $8y - 9$

   In g through i, arrange the polynomial in descending order.

   g) $5x^2 + x^3 + 2x - 1$ h) $x^4 - 9 + 6x^2$

   i) $x^2 + 3x^3 - 1$

2. a) Simplify the polynomial:
   $2x^2 + 10x - 3 + 5x^2 - 7x + 8$

   b) Add:

$$\begin{array}{rrrr} 2x + 7 & 9xy + 6ab & x^2 + 15x - 3 & 5x^2 - 4x + 3 \\ \underline{6x - 8} & \underline{2xy - 4ab} & \underline{6x^2 - 8x - 5} & x^2 \qquad + 4 \\ & & & \underline{3x^2 - x - 5} \end{array}$$

   c) Find the sum of $5x^2 - 7x + 3$ and $8x - 5 - 3x^2$.

3. Find the perimeter of a triangle whose sides are:
   $3x^2 + 2x - 1$, $5x + 3$, $x^2 - 8x + 2$

4. a) Subtract and check the result:
From $x^2 + 5x - 2$, subtract $x^2 + 9x + 13$.

b) How much greater than $3x^2 + 7x - 2$ is $5x^2 - 11x - 5$?

c) Subtract:
$$\begin{array}{r} 5x - 10 \\ \underline{3x + 8} \end{array}$$

d) Subtract:
$$\begin{array}{r} 3y^2 + 8y - 10 \\ \underline{y^2 - 6y + 2} \end{array}$$

e) Subtract:
$$\begin{array}{r} x^2 \qquad - 9 \\ \underline{-5x^2 + 6x + 2} \end{array}$$

f) Subtract:
$$\begin{array}{r} 3x^2 + 5x - 7 \\ \underline{- 4x + 3} \end{array}$$

SUMMARY:

1. What procedure do we follow to find the sum of polynomials?
   What is the sum of $5x^2 + 3x + 2$ and $x^2 - 2x - 5$?
2. What is the procedure to find the difference of polynomials? What is the difference between $3x^2 + 1$ and $x^2 + x - 3$?
3. What is the most important thing to remember when adding or subtracting polynomials?

HOMEWORK:

1. Add:

a)
$$\begin{array}{r} -7x^2 - 1 \\ \underline{3x^2 + 8} \end{array}$$

b)
$$\begin{array}{r} 2x + 9y \\ \underline{x - 5y} \end{array}$$

c)
$$\begin{array}{r} x^2 + 3x - 2 \\ \underline{4x^2 - 7x + 10} \end{array}$$

d)
$$\begin{array}{r} -4x^2 + 6x + 11 \\ \underline{x^2 + 4x - 7} \end{array}$$

e)
$$\begin{array}{r} 3x^2 + 3x - 2 \\ \underline{x^2 - x + 10} \end{array}$$

2. Subtract:

a)
$$\begin{array}{r} 9x^2 - 3x - 8 \\ \underline{x^2 + x - 2} \end{array}$$

b)
$$\begin{array}{r} 5x + 4 \\ \underline{3x - 10} \end{array}$$

c)
$$\begin{array}{r} 10x^2 + x - 2 \\ \underline{7x^2 + 7x - 9} \end{array}$$

d)
$$\begin{array}{r} 5x^2 + 9x + 6 \\ \underline{3x^2 - x - 7} \end{array}$$

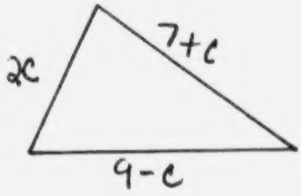
e)
$$\begin{array}{r} 8x^2 + 4x - 3 \\ \underline{x^2 - 5x - 2} \end{array}$$

3. From $6x^2 - 7x + 3$, subtract $4x^2 + 3x - 8$.

4. Subtract $x^2 + 6x - 2$ from $3x^2 - 5x + 3$.

5. Find the perimeter of a triangle whose sides are:
   a) $2x^2 - 5x + 8$, $3x^2 + 6x - 2$, $x^2 + 3x + 4$
   b) $6x - 5$, $3x + 1$, $2x - 4$

*6. For what values of c will the triangle have two sides with the same length?



Lesson 24 - Multiplication Polymonial by Monomial

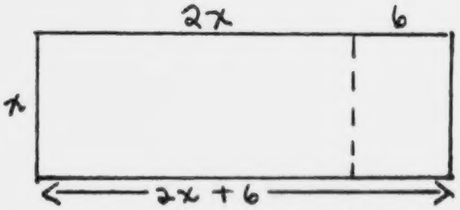
AIM: How do we multiply a polynomial by a monomial?

PERFORMANCE OBJECTIVES: The students will be able to...

1. Recall the distributive principle.
2. Multiply a polynomial by a monomial.

VOCABULARY: distributive property

MOTIVATION:



What is the area of a rectangle whose length is $2x + 6$ and whose width is x?
(Hint: Try separating it into two rectangles such that the length of one rectangle is $2x$, and the length of the other is 6.)

DEVELOPMENT (Include pivotal questions):

1. We know that the distributive property states that
   $a(b + c) = ab + ac$
   Therefore, $x(2x + 6) = x(2x) + x(6)$
   $2x^2 + 6x$
   This can be shown geometrically by the area problem (see above). Area = length x width
   $= x(2x + 6) = 2x^2 + 6x$
   Use the distributive property to find the product:
   $8(2x + 3y) = 8(2x) + 8(3y) = 16x + 24y$
2. Elicit the following: To multiply a polynomial by a monomial, we use the distributive property of multiplication over addition. Each term within the

parentheses must be multiplied by the monomial preceding (or following) the parentheses.

The multiplication may also be arranged vertically:

$$\begin{array}{r} 2x - 7y \\ 5 \\ \hline 10x - 35y \end{array}$$

3. Do application 1.

4. If the monomial is a negative number or has a negative coefficient, be especially careful with signs.

   $-5(x - 4) = -5x + 20$

5. Do application 2.

6. The distributive property could also be written as

   $(b + c)a = ba + ca$

   Apply the above distributive principle to examples:

   $(2x + 6)(4x) =$

   $(3x - 5y)(-4) =$

7. Have students realize that the product of a polynomial and -1 is the additive inverse (opposite) of the polynomial.

   $-1(3x^2 + 2) = -3x^2 - 2$

APPLICATIONS:

1. In exercises a through d, compute the product:

   a) $6x^2(2x - 3xy)$ b) $4(3a + 2d)$

   c) $5x(3x + 4)$ d) $2x(x - 8)$

   e) Compute the product: $3a^2b(2a - 5ab + ab^2)$

   f) Compute the product: $xy(x + y)$

   g) Multiply to remove parentheses: $3d(d^2 - 2d)$

   h) Compute the product: $7r^2s^2(-3r^2 + 2rs - 5s^2)$

   i) Find the area of a rectangle with length $3x$ and width $5x^2 - 8x - 9$.

2. Simplify:

   a) $-4(3x - 5y)$ b) $-2a(3a - a^2)$

   c) $-6(2x^2 - 5x - 4)$ d) $-rs(6r^4s - 3s^3)$

   e) $-4a^2(12a - 4a^2)$ f) $-(3x^2 + 5y)+(7x^2 - 9y)$

SUMMARY:

1. Find the error: $-4(x + 3) = -4x + 12$.
2. What is the procedure for multiplying a polynomial by a monomial?
3. Think of a polynomial. Multiply it by negative one. Add the result to the original polynomial. What is the sum?

HOMEWORK:

1. Compute:

   a) $3(x^2 + 7x)$
   b) $2(5x + 1)$
   c) $4x(x^2 + 6x - 5)$
   d) $7(2x^3 + x^2 + 10)$
   e) $8x(5x - 2)$
   f) $-5(4x + 6)$
   g) $xy(3x^2y + 2xy^2 + 4y^3)$
   h) $4(2x - 5y + 10)$
   i) $2x^2(x^3 - 9x^2 + 4x + 5)$
   j) $-3x(x^2 - 2x + 9)$
   k) $(2x - 5)(-3)$

2. Find the area of a rectangle having dimensions $2x$ and $5x^2 + 7x - 1$.
3. Compute: $5a^2b^2(-5a^2 + 2ab - 4b^2)$ and check by letting $a = 2$ and $b = 3$.

Lesson 25 - Division of Polynomial by Monomial

AIM: How do we divide a polynomial by a monomial?

PERFORMANCE OBJECTIVES: The students will be able to...

1. Recall the rules for division of monomials.
2. Divide a polynomial by a monomial.
3. Divide a polynomial by a monomial where one term of the polynomial is the same as the monomial.
4. Check the quotient.

VOCABULARY: divisor, dividend, quotient

MOTIVATION:

The total cost of a fishing trip is 4x + 4y dollars. Four boys are going on the trip. How much should each pay?

DEVELOPMENT (Include pivotal questions):

1. Dividing by 4 means the same as multiplying by $\frac{1}{4}$. Therefore,

   $$\frac{4x+4y}{4} = \frac{1}{4}(4x + 4y) = \frac{1}{4}(4x) + \frac{1}{4}(4y) = \frac{4x}{4} + \frac{4y}{4} = x + y$$

   This means that x + y can be obtained by dividing each term of 4x + 4y by 4:

   $$\frac{4x+4y}{4} = \frac{4x}{4} + \frac{4y}{4} = x + y$$

2. To divide a polynomial by a monomial, divide each term of the polynomial by the monomial.
   Recall the rules for dividing monomials.

Examples:

$$\frac{6a^4 - 10a^5}{2a^2} = \frac{6a^4}{2a^2} + \frac{-10a^5}{2a^2} = 3a^2 - 5a^3$$

$$\frac{18x^4 + 12x^3}{6x^2} = 3x^2 + 2x$$

To check, multiply: $6x^2(3x^2 + 2x)$
(the divisor by the quotient).

The answer should be the dividend: $18x^4 + 12x^3$.

3. Do application 1.

4. What is the answer when a number is divided by itself? What happens when a term in the dividend is equal to the divisor? (Answer: a term in the quotient equals one.)

   Example:

$$\frac{x^2 + x}{x} = \frac{x^2}{x} + \frac{x}{x} = x + 1 \quad (x \neq 0)$$

$$\frac{p+prt}{p} = 1 + rt \quad (p \neq 0)$$

5. Do application 2.

APPLICATIONS:

1. Divide and check:

   a) $\frac{8x + 6y}{2}$ b) $\frac{15x^3 - 10x^2}{5x}$

   c) $\frac{16y^5 - 8y^4}{4y^2}$ d) $\frac{9x^7 - 6x^5}{-3x^2}$

   e) $\frac{4x^5 + 12x - 20}{-4}$ f) $\frac{2rs^2 - 5r^2s}{rs}$

   g) $\frac{x^2 + 7x}{x}$ h) $\frac{15x^3 - 10x^2}{5x}$

2. Divide and check:

   a) $\frac{14a^3 - 7a^2}{7a^2}$ b) $\frac{9a^3 - 6a^2 + 3a}{3a}$

Compute the quotient:

c) $\dfrac{24x^3y^4 - 12x^2y^2 - 4xy}{-4xy}$

d) $\dfrac{12A + 3B - 3}{-3}$

SUMMARY:

1. Explain the rule for dividing a polynomial by a monomial.
2. Explain the error: $\dfrac{18x^4 + 6}{6} = 3x^4 + 6$
3. Find the width of a rectangle whose area is $16x^2 - 12x$ and whose length is $4x$.

HOMEWORK:

Simplify and check (1 - 7):

1. $\dfrac{3x^2 + 12x + 21}{3}$
2. $\dfrac{2x^3 - 8x^2 + 32x}{2x}$
3. $\dfrac{8x+16}{4}$
4. $\dfrac{18x^3 + 30x^2 - 36x}{3x}$
5. $\dfrac{10x^2 + 5x}{x}$
6. $\dfrac{8x^2 - 16x + 4}{4}$
7. $\dfrac{6x^3y^2 - 12x^2y^2 + 3xy}{3xy}$
8. Find the length of a side of a square whose area is $9x^2$.
9. Find the length of a rectangle whose area is $25x^3 - 10x^2 + 5x$ and whose width is $5x$.
10. The perimeter of a square is represented by $12x - 4$. Express the length of one side of the square in terms of x.

Lesson 26 - Translating English Sentences into Algebraic Expressions

AIM: How do we translate an English sentence into an algebraic expression?

PERFORMANCE OBJECTIVES: The students will be able to...

1. Use a variable to represent the unknown quantity.
2. Translate the sentence into an equation or inequality.
3. Distinguish between an equation and an inequality.

VOCABULARY: equation, variable, inequality, increased by, decreased by, diminished by, more than, less than, quotient, product, difference

MOTIVATION:

1. In a trial, a judge refused to accept evidence which was "hearsay". Why was this less valid than "eyewitness" testimony?
2. Six times a number is 492. If x represents the number, which of the following sentences correctly expresses this relationship?

   a) $x = 6(492)$ b) $x = \frac{6}{492}$ c) $6x = 492$ d) $6 = 492x$

DEVELOPMENT (Include pivotal questions):

1. Discuss motivation. "Hearsay" information is less acceptable because distortions may have been introduced as the story was communicated through a chain of listeners, each of whom rephrased the facts. In translating mathematical sentences from English to symbolic notation, students must be careful to represent all of the included information in an accurate and unambiguous manner.

2. Verbal sentences can be translated into equations. An *equation* is a sentence that uses the symbol = to state that two expressions are equal. We use a letter called a *variable* to represent or hold the place for a number. For example,

   three times a number equals 21: $3n = 21$
   a number increase by 12 equals 17: $x + 12 = 17$
   a number decreased by 4 equals 5: $y - 4 = 5$
   a number divided by 2 equals 10: $\frac{x}{2} = 10$

   To write a verbal sentence as an equation, choose a letter (our variable) to represent a number. Then proceed to use the letter, with arithmetic operations, to express the verbal sentence as an equation.

3. Do application 1.

4. Some verbal sentences and equations involve two operations. For example, two times a number decreased by 3 equals 13. Let n equal the number ($2n - 3 = 13$).

5. Do application 2.

6. An *inequality* is a statement that one quantity is *not* equal to another. What are three symbols of inequality? Review the meaning of the symbols of inequality: $\neq$ (not equal to), $>$ (greater than), $<$ (less than).
   For example, the sum of 8 and 10 is not equal to the product of 8 and 2.

   $8 + 10 \neq 8 \times 2$

   28 is greater than 21

   $28 > 21$

   15 + 9 is less than 30. $15 + 9 < 30$
   10 diminished by 2 is greater than 5. $10 - 2 > 5$

7. Do application 3.

8. Determine by inspection the solution of open sentences written in application 1.
   a) $x + 7 = 15$
   b) $x - 5 = 3$
   c) $4x = 24$
   d) $x - 6 = 1$
   e) $\frac{x}{5} = 7$
   f) $x - 12 = 88$

9. If the domain of each variable is an integer between -10 and 10 inclusive, solve:
   a) $x \not> 0$ (Note: "$\not>$" means is not greater than.)
   b) $2y + 1 \not< 0$
   c) $(2y + 1) \not< 0 \lor (7 - y) > 0$

APPLICATIONS:

1. Translate each of the following sentences into equations.
   a) Seven more than a number is 15.
   b) Five less than a number is 3.
   c) Four times a number is 24.
   d) A number decreased by 6 is 1.
   e) A number divided by 5 is 7.
   f) If 12 is subtracted from a number, the result is 88.

2. Translate each of the following sentences into equations.
   a) Twice a number increased by 3 is 23.
   b) One half a number decreased by 2 is 5.
   c) The sum of four times a number and 3 equals 35.

3. Translate each of the following sentences into inequalities.
   a) 40 increased by a number is less than 80.
   b) A number decreased by 9 is greater than 11.
   c) The product of 7 and 4 is not equal to 21.
   d) If 6 is subtracted from 8 times a number, the result is greater than 26.
   e) 3 less than two times a number is more than 11.

SUMMARY:

1. What are the advantages of having a mathematical sentence in algebraic notation?

2. Make up verbal problems that could be solved using the equations:
   a) $2x + 3 = 5$
   b) $4n = 12$

3. Explain why the answer to a verbal problem should not be checked only in the equation.

4. What are 3 symbols of inequality?

HOMEWORK:

Write an equation or inequality for each of the following sentences:

1. Twice a number decreased by 9 is 7.
2. A number increased by 11 is 33.
3. Three times a number is 54.
4. The sum of three times a number and 4 is 19.
5. A number divided by 5 is 11 more than the number.
6. Four times a number decreased by 9 is less than 31.
7. The product of 10 and 3 is not equal to 32.
8. Twice a number increased by 3 is more than 9.
9. A number decreased by 2 is greater than 7.
10. If 17 is subtracted from 4 times a number, the result is less than 3.
11. Compute the quotient and check:

    a) $\frac{9x^2 - 6}{3}$ b) $\frac{9y^2 - 6y}{3y}$

Lesson 27 - Solving eq. $x \pm a = b$

AIM: How do we solve an equation of the type $x + a = b$?

PERFORMANCE OBJECTIVES: The students will be able to...

1. Solve equations of the type $x + a = b$.
2. Check the solutions.
3. State what is meant by solving an equation.

VOCABULARY: variable, equation, inverse operation, additive inverse, equivalent equation, replacement set, solution set, root

MOTIVATION:

Using $\{1,2,3,4,5,6,7\}$ as the replacement set (domain), find the solution set for each of the following equations:

$x + 3 = 10$

$x - 3 = 4$

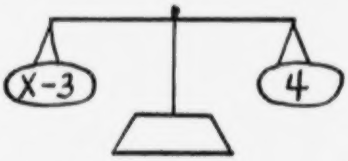
$9 = x + 2$

What do the solution sets have in common?

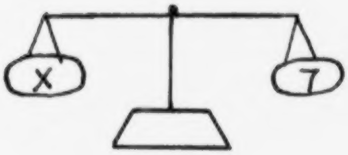
DEVELOPMENT (Include pivotal questions):

1. Refer to motivation: Elicit from the class that for each of the equations the solution set is $\{7\}$. Such equations are said to be equivalent. Have students note that a given equation may have many equivalent equations.
   If the replacement set is large, what disadvantage is there in using the replacement method to solve an equation?
   Is there a better method for solving an equation?

2. Think of an equation as a scale in balance.

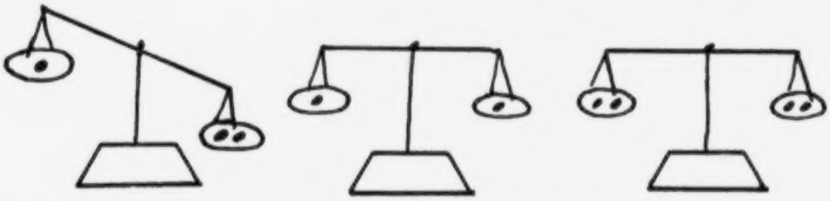


Suppose we have the equation x - 3 = 4 to solve. We want to get the variable alone on one side of the equation. How do we do it? Add +3 to both sides.



$$\begin{aligned} x - 3 &= 4 \\ x - 3 + (+3) &= 4 + (+3) \\ x + 0 &= 7 \\ x &= 7 \end{aligned}$$

3. Study the three scales. If we add a weight to only one side, then the scale is no longer in balance. However, if we add the same weight to both sides, then the scale will balance. Think of an equation as a scale in balance.



Study the three scales. If we subtract weight from only one side, then the scale is no longer balanced. However, if we subtract the same weight from both sides, then the scale will balance. Think of an equation as a scale in balance.

Review subtraction as the addition of the additive inverse. Elicit that subtracting a number is the same as adding the opposite or additive inverse.

What is meant by the additive inverse of a number? What is the additive inverse of zero? Which number is the additive inverse of each of the following: 3, -3, 2?

Elicit: "If the same number is added to both members of an equation, the equation remains in balance and an equivalent equation results."

4. Demonstrate this property with numerical expressions.

$$\begin{aligned} 7 &= 7 \\ 7 + 2 &= 7 + 2 \\ 9 &= 9 \end{aligned} \qquad \begin{aligned} 7 &= 7 \\ 7 + (-2) &= 7 + (-2) \\ 5 &= 5 \end{aligned}$$

Demonstrate how the additive inverse can be used to solve an equation.

For example: $x - 2 = 5$

$$\begin{aligned} x - 2 + (+2) &= 5 + (+2) \\ x &= 7 \end{aligned}$$

$x = 7$ is a trivial equation equivalent to $x - 2 = 5$, the original equation.

Using the above method, have the class solve the motivation examples:

$$\begin{aligned} x + 3 &= 10 \\ x + 3 + (-3) &= 10 + (-3) \\ x &= 7 \end{aligned} \qquad \begin{aligned} x - 3 &= 4 \\ x - 3 + (+3) &= 4 + (+3) \\ x &= 7 \end{aligned}$$

$$\begin{aligned} 9 &= x + 2 \\ 9 + (-2) &= x + 2 + (-2) \\ 7 &= x \end{aligned}$$

5. Do application 1.

6. How do we know that the answer is correct?

The solution can be checked by substituting the value 7 for the original equation.

Check:

$$\begin{aligned} x + 3 &= 10 \\ 7 + 3 &\overset{?}{=} 10 \\ 10 &= 10 \checkmark \end{aligned} \qquad \begin{aligned} x - 3 &= 4 \\ 7 - 3 &\overset{?}{=} 4 \\ 4 &= 4 \checkmark \end{aligned} \qquad \begin{aligned} 9 &= x + 2 \\ 9 &\overset{?}{=} 7 + 2 \\ 9 &= 9 \checkmark \end{aligned}$$

The purpose of a check is to verify that there are no computation errors.

Similarly, to show 7 is the root of the equation $x + 3 = 10$, substitute the value 7 for x in the original equation and see if the resulting equality is true.

To solve an equation means to find its solution set. Equations that have the same solution set are called equivalent equations.

Thus, "solve for the variable", "find the roots", and "find the solution set" are equivalent statements.

7. Consider more difficult equations using signed numbers.

solve for x	solve for z
$5 + x = -9$	$-27 = z - 20$
$\underline{-5 \qquad = -5}$	$\underline{+20 = \qquad + 20}$
$x = -14$	$-7 = z$

Check:

$5 + x = 9$	$-27 = z - 20$
$5 + (-14) \overset{?}{=} -9$	$-27 \overset{?}{=} -7 - 20$
$-9 \overset{?}{=} -9$ ✓	$-27 =^{?} -27$ ✓

Review set notation.

Consider the equation $x + 3 = 14$. The solution set is 11 . This is sometimes written as $\{x \mid x + 3 = 14\} = \{11\}$ and is read as "the set of all numbers x such that x + 3 = 14 is the set consisting of the number 11".

8. Do applications 2 and 3.

9. Help students develop skill in solving problems. Help students think through and express relationships involving number problems.

Examples:

One number is 6. Another is 14 more than the first. What is the second number? What is the sum of the numbers? (Answers: 20, 26)

In a class election, Juliana received 14 more votes than Karen received. If Juliana received 20 votes, how many did Karen receive?

Let x = number of votes Karen received

x + 14 = number of votes Juliana received

$$\begin{array}{rcr} x + 14 & = & 20 \\ \underline{\quad -14} & & \underline{-14} \\ x & = & 6 \end{array}$$

Check: Is 20 fourteen more than 6? (14 + 6 = 20)

How do we check a verbal problem?

10. Do application 4.

MATERIALS: a double-pan balance, weights

APPLICATIONS:

1. Solve for x:
   a) $x + 2 = 5$ b) $12 = x - 91$
   c) $3 = x + 5$ d) $10 + x = 7$
   e) $x - 9 = 63$ f) $8 = x - 11$
2. Find the solution set and check:
   a) $2 + x = -7$ b) $-4 = -6 + q$
   c) $r + (-3) = 4$ d) $-32 = 16 + s$
   e) $\{x \mid x - 20 = 43\}$ f) $\{y \mid y - 20 = -6\}$
   g) $-27 = z - 20$ h) $x + 6 = 6$
3. Find the root of the equation: $-14 + x = 23$
   Is 4 a root of the equation: $11 = x - (-7)$?
4. After Jack lost 22 pounds, he weighed 179 pounds. Find Jack's original weight. (Use an equation to solve.)

SUMMARY:

1. What is meant by equivalent equations?
2. What method do we use to tell whether two equations are equivalent?
3. How many equations can we find which are equivalent to a given equation?
4. How are the equations related when the same number is added to both members of an equation to form a new equation?
5. What is the advantage of using the additive inverse over the trial and error method of solution?
6. How do we determine what additive inverse to use to solve an equation?

HOMEWORK:

1. Is -3 a root of the equation $-12 = x - 9$?

2. Solve for the variable and check:

   a) $x - 5 = 8$ b) $x + 71 = 42$ c) $31 = y - 10$

3. Find the solution set and check:

   a) $x + 7 = 4$ b) $8 + x = 17$

   c) $\{x \mid x - 7 = -2\}$ d) $\{x \mid 10 = -7 + x\}$

4. Find the root of each of the following equations:

   a) $x - 27 = -12$ b) $26 + x = -4$

   c) $-8 = x - 395$ d) $-6 + x = -16$

5. Seven years from now, Mark will be 19 years old. How old is Mark now?

6. A number decreased by 5 is 19. Find the number.

7. Compute the product and check by letting $x = 2$, $y = 3$.

   $$2xy^2(3x - 4y)$$

8. Compute the quotient and check by letting $r = 2$ and $d = 3$.

   $$\frac{12rs^2 - 9sr^2}{3rs}$$

Lesson 28 - Solving eq. ax = b

AIM: How do we solve an equation of the type ax = b?

PERFORMANCE OBJECTIVES: The students will be able to...

1. Translate verbal sentences to equations of the type ax = b.
2. Solve equations of the type ax = b where a and b are rational.

VOCABULARY: multiplicative inverse, reciprocal, solution set, root

MOTIVATION:

Using $\{0,1,2,3,4\}$ as the replacement set, find the solution set for each of the following equations:

a) $2x = 6$ b) $x = 3$

c) $\frac{1}{3}x = 1$ d) $-6x = -18$

DEVELOPMENT (Include pivotal questions):

1. Refer to the motivation.

   Why are the equations equivalent?

   Answer: For each of the equations the solution set is $\{3\}$.

   Consider solving the equation 2x = 6. We want to get the variable alone on one side of the equation. How do we do it? Can we use the strategy learned yesterday (adding the opposite on each side)? Why isn't this method appropriate? What is the desired coefficient of x? By what number should we multiply 2 to obtain positive one as the product?

   $$2x = 6$$
   $$(\tfrac{1}{2})2x = (\tfrac{1}{2})6 \qquad \text{(Multiply by } \tfrac{1}{2}.\text{)}$$
   $$x = 3$$

Similarly, ask: What number times $\frac{1}{3}$ will give us the coefficeint 1?

$$\text{If } \frac{1}{3}x = 1$$

$$\frac{x}{3} = 1$$

$$3 \cdot \frac{x}{3} = 3 \cdot 1 \quad \text{(Multiply both members}$$

$$x = 3 \quad \text{of equation by 3.)}$$

2. What name is given to a pair of numbers whose product is +1?

   What number is the multiplicative inverse of each of the following?

   $$2,\ 3,\ -6,\ \frac{1}{5},\ \frac{1}{3},\ -\frac{2}{5}$$

   Demonstrate how the multiplicative inverse can be used to solve an equation. Remember that the product of a number and its multiplicative inverse (reciprocal) is one.

   For example, refer to the motivation: $-6x = -18$. What is the multiplicative inverse of -6? (Answer: $-\frac{1}{6}$)

   $$(\frac{1}{6})(-6x) = (-\frac{1}{6})(-18)$$

   $$x = \frac{-18}{-6}$$

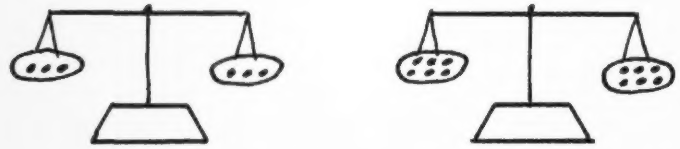
   $$x = +3$$

   Elicit from the class:

   Rule: "If both sides of an equation are multiplied by the same non-zero number (multiplicative inverse), the resulting equation will have the same solution as the original."

   Thus: If $a = b$, then $ac = bc$. $(c \neq 0)$

3. Using an approach similar to that of the preceding lesson, the double pan balance could be used to reinforce the new concept.



If the weight on each pan is doubled, the scale will remain in balance.

4. How do we know the answer is correct?

The solution can be checked by substituting for x the value of 3 in the original equation. (Apply to motivation problems.)

Check:

$$\begin{array}{cccc} 2x = 6 & x = 3 & \frac{1}{3}(x) = 1 & -6x = -18 \\ 2(3) \overset{?}{=} 6 & 3 \overset{?}{=} 3 & \frac{1}{3}(3) \overset{?}{=} 1 & -6(3) \overset{?}{=} -18 \\ 6 = 6 \checkmark & & 1 = 1 \checkmark & -18 = -18 \checkmark \end{array}$$

5. Do applications 1 and 2.

6. Consider more difficult equations using signed numbers.

a) Solve for x:

$$\begin{aligned} \tfrac{2}{3}x &= -4 \\ (\tfrac{3}{2})\tfrac{2}{3}x &= (\tfrac{3}{2})(-4) \\ x &= \frac{-12}{2} \\ x &= -6 \end{aligned}$$

b) Find the roots of the following equations.

1)
$$\begin{aligned} -\tfrac{4}{5}x &= 12 \\ (-\tfrac{5}{4})(-\tfrac{4}{5}x) &= (-\tfrac{5}{4})12 \\ x &= \frac{-60}{4} \\ x &= -15 \end{aligned}$$

2)
$$\begin{aligned} \tfrac{x}{5} &= \tfrac{1}{3} \\ (\tfrac{5}{1})\tfrac{x}{5} &= (\tfrac{5}{1})\tfrac{1}{3} \\ x &= \tfrac{5}{3} \\ x &= 1\tfrac{2}{3} \end{aligned}$$

c) Solve for x: $0.02x = 32.6$

Ask: What is the reciprocal of .02?

Since $.02 = \frac{2}{100}$ or $\frac{1}{50}$, its reciprocal is 50.

the class should also consider an alternate strategy for handling equations with decimal coefficients: multiply each side by a power of ten sufficient to make all coefficients integers.

$$\begin{aligned} (100)(.02x) &= (100)32.6 \\ 2x &= 3260 \\ (\tfrac{1}{2})2x &= (\tfrac{1}{2})3260 \\ x &= 1630 \end{aligned}$$

d) Find x: $-12 = \frac{2}{5}x$

e) Solve for x: $-x = 3$

$(-1)(-x) = (-1)(3)$

$x = -3$

f) (Review set notation.) Find the solution set if $x \in \{\text{signed numbers}\}$:

$\{x \mid 3x = -15\}$

$3x = -15$

$(\frac{1}{3})3x = (\frac{1}{3})(-15)$

$x = \frac{-15}{3}$ Solution set $= \{-5\}$ (answer)

$x = -5$

7. Do applications 3 and 4.

8. Help pupils develop skill in solving problems.
Sal scored twice as many points as Sam. If Sal scored 16 points, how many points did Sam Score?

Let x = the number of points Sam scored
2x = the number of points Sal scored

$2x = 16$

$(\frac{1}{2})2x = (\frac{1}{2})16$

$x = 8$

9. Do application 5.

MATERIALS: Double pan balance, weights

APPLICATIONS:

1. Find the multiplicative inverse of each of the following:

$5,\ \frac{1}{2},\ 8,\ -6,\ -\frac{1}{2},\ -\frac{2}{3},\ 4,\ -1,\ -\frac{1}{8}$

2. Solve for the variable and check:

a) $6x = 12$ b) $\frac{1}{2}x = 15$
c) $8x = 56$ d) $5x = 20$
e) $\frac{2}{3}x = 8$

3. Find the solution set and check:
   - a) $-6x = -12$
   - b) $-\frac{1}{2}x = 12$
   - c) $8 = -\frac{2}{3}x$
   - d) $5 = \frac{x}{9}$
   - e) $4x = -20$
   - f) $-x = 18$
   - g) $-\frac{1}{8}x = -1$
   - h) $72 = -72x$

4. Find the root of each of the following equations:

$$-\frac{3}{8}x = -1, \quad 6x = -24, \quad \frac{2}{5}x = 6$$

5. a) The Jets scored twice as many points as the Giants. If the Jets scored 24 points, how many points did the Giants score?

   b) The perimeter of a square is 72 centimeters. Find the measure of the side of a square.

SUMMARY:

1. How are the equations related when both members of an equation are multiplied by the same nonzero number to form a new equation?

2. Why must zero be excluded when using the multiplicative inverse?

3. What is the advantage of using the multiplicative inverse over the trial and error method of solution?

4. How do we determine the multiplicative inverse to use to solve an equation of the form $ax = b$?

5. If $3x = 27$, find $2x$.

6. If the replacement set is large, what disadvantage is there in using the replacement method to solve an equation?

HOMEWORK:

1. Solve and check:
   - a) $7x = 42$
   - b) $3c = 12$
   - c) $18 = 6x$
   - d) $0.03y = 36$
   - e) $-\frac{2}{5}x = 4$
   - f) $-\frac{2}{3}x = 8$

g) $-\frac{3}{4}x = -90$ h) $-x = 7$

2. Find the solution set and check:
   - a) $-5x = 30$ b) $3x = -48$
   - c) $-6x = -36$ d) $27 = -\frac{a}{9}$
   - e) $\frac{c}{4} = -16$ f) $-\frac{2}{7}y = 6$

3. The area of a rectangle 5 feet wide is 80 square feet. Find the length of the rectangle.

4. Compute the product and check:

$$-2ab^2(3ac - 4b)$$

5. Compute the quotient and check:

$$\frac{-36r + 48rs^2}{-6r}$$

6. Solve: $|x - 3| < 7$ where $x \in \{\text{Integers}\}$.

7. If both sides of an equation are multiplied by zero, will the resulting equation be equivalent to the original one?

Lesson 29 - Solving eq. $ax \pm b = c$

AIM: How do we solve equations of the type $ax + b = c$?

PERFORMANCE OBJECTIVES: The students will be able to...

1. Recall the meaning of "solving an equation".
2. Isolate the variable using properties of inverses and identities.
3. Check the solution.
4. Solve equations of the type $ax + b = c$ where a, b and c are rational.

VOCABULARY: No new terms

MOTIVATION:

Problem: Twice the number of people invited came to the party. Four left early and eight stayed until the end. How many people were invited?
Solution: Step One
Translate into an equation: Twice a number decreased by four equals eight.

DEVELOPMENT (Include pivotal questions):

1. Refer to the motivation. Elicit the equation: $2x - 4 = 8$. How can we find the solution?
   Develop alternate approaches:
   Method I:
   What equivalent equation would be obtained by adding the additive inverse of -4 to both members of the equation $2x - 4 = 8$?

$$\begin{array}{rcr} 2x - 4 & = & 8 \\ +4 & = & +4 \\ \hline 2x & = & 12 \end{array}$$

What equivalent equation would be obtained by applying the multiplicative inverse of 2 to both members of the equation $2x = 12$?

$$\begin{array}{rcl} 2x & = & 12 \\ (\frac{1}{2})\,2x & = & (\frac{1}{2})\,12 \\ x & = & 6 \end{array} \qquad \text{Solution set} = \{6\}$$

Method II:

What equivalent equation would be obtained by applying the multiplicative inverse to both members of the equation $2x - 4 = 8$?

$$\begin{array}{rcl} 2x - 4 & = & 8 \\ (\frac{1}{2})(2x - 4) & = & (\frac{1}{2})\,8 \\ x - 2 & = & 4 \end{array}$$

What equivalent equation would be obtained by applying the additive inverse to both members of the equation $x - 2 = 4$?

$$\begin{array}{rcr} x - 2 & = & 4 \\ +2 & = & +2 \\ \hline x & = & 6 \end{array} \qquad \text{Solution Set} = \{6\}$$

Elicit that we used two properties of inverses to solve the equation $2x - 4 = 8$.

Comparing Method I and Method II, the solution set 6 was the same.

Check that 6 satisfies the conditions of the given problem. Which of these procedures seems easier?

2. The student often fails to operate on every term of the equation if he uses the multiplicative inverse first. Also the arithmetic is often more complicated. Therefore, the student should first use the additive inverse.

   Consider $4x - 7 = 17$.

According to Method I

Use additive inverse.

$$\begin{array}{rcl} 4x - 7 & = & 17 \\ +7 & = & +7 \\ \hline 4x & = & 24 \end{array}$$

Use multiplicative inverse.

$$4x = 24 \longrightarrow (\tfrac{1}{4})4x = (\tfrac{1}{4})24$$

$x = 6$ Solution Set = $\{6\}$

According to Method II

Use multiplicative inverse

$$4x - 7 = 17$$

$$(\tfrac{1}{4})(4x - 7) = (\tfrac{1}{4})17$$

$$\frac{4x-7}{4} = \frac{17}{4}$$

$$\frac{4x}{4} - \frac{7}{4} = \frac{17}{4}$$

$$x - \frac{7}{4} = \frac{17}{4}$$

Use additive inverse.

$$\begin{array}{rcl} x - \frac{7}{4} & = & \frac{17}{4} \\ + \frac{7}{4} & = & \frac{17}{4} \\ \hline x & = & \frac{24}{4} \end{array}$$

$x = 6$ Solution set = $\{6\}$

Elicit that in each procedure, the solution set of $4x - 7 = 17$ is $\{6\}$.

Which of these procedures seems easier? Elicit that using the additive inverse first, and then the multiplicative inverse, is more efficient.

Have students check the solution.

Check

$$4x - 7 \overset{?}{=} 17$$

$$4(6) - 7 \overset{?}{=} 17$$

$$24 - 7 \overset{?}{=} 17$$

$$17 = 17 ✓$$

3. Do application 1.

4. Consider more difficult equations. Solve for the variable and check:

$$\begin{array}{rcl} 10 - 2x & = & -4 \\ -10 & = & -10 \\ \hline -2x & = & -14 \\ (-\frac{1}{2})(-2x) & = & (-\frac{1}{2})(-14) \\ x & = & 7 \end{array}$$

Solution Set = $\{7\}$

Check

$$\begin{array}{rcl} 10 - 2x & \overset{?}{=} & -4 \\ 10 - 2(7) & \overset{?}{=} & -4 \\ 10 - 14 & \overset{?}{=} & -4 \\ -4 & \overset{?}{=} & -4 \checkmark \end{array}$$

Review set notation. Find the solution set if $x \in \{\text{signed numbers}\}$: $\{x \mid 3x - 5 = 16\}$

$$\begin{array}{rcl} 3x - 5 & = & 16 \\ +5 & = & +5 \\ \hline 3x & = & 21 \\ (\frac{1}{3})3x & = & (\frac{1}{3})21 \\ x & = & 7 \end{array}$$

Solution Set = $\{7\}$

Solve for x: $0.2x + 0.3 = 8.1$ (As yesterday, multiply by 10 to obtain integer coefficients, then proceed.)

$$\begin{array}{rcl} 2x + 3 & = & 81 \\ -3 & = & -3 \\ \hline 2x & = & 78 \\ (\frac{1}{2})2x & = & (\frac{1}{2})78 \\ x & = & 39 \end{array}$$

Solution Set = $\{39\}$

5. Do application 2.

6. Solve verbal problem:

Linda's father weighs 20 pounds less than twice what Linda weighs. If her father weighs 180 pounds, how much does Linda weigh?

Let x = Linda's weight

2x - 20 = Linda's father's weight

$$\begin{array}{rcl} 2x - 20 & = & 180 \\ +\ 20 & = & 20 \\ \hline 2x & = & 200 \\ (\frac{1}{2})2x & = & (\frac{1}{2})200 \\ x & = & 100 \end{array}$$

Linda weighs 100 lbs.

Check: 180 pounds is 20 pounds less than twice 100.

$$180 = 2(100) - 20$$
$$180 = 200 - 20$$
$$180 = 180$$

7. Do application 3.

APPLICATIONS:

1. a) Solve for c: $\frac{1}{3}c + 2 = 4$
   b) Solve for x: $4x - 3 = 41$
   c) Solve for x: $11x + 12 = 89$

2. Solve and check:
   a) $-6c + 30 = -6$ b) $-36 = 10y + 14$
   c) $10c - 90 = -9$ d) $3a + 0.2 = 5$
   e) $73 - 20x = -27$ f) $-\frac{3}{4}x - 3 = 6$
   g) $7 - \frac{2}{5}x = 15$

3. a) Thirteen is 2 less than five times n. Find n.
   b) The length of a room is four feet less than three times the width. If the length is 26 feet, find the width of the room.

SUMMARY:

1. Why is the use of just one inverse operation not always adequate to solve an equation?

2. If you changed the order in which you used the operations needed to solve an equation of the form $ax + b = c$, how would the results compare? Explain.

3. The equation $5x + 10 = 55$ has the same solution set as the equation
   a) $x = 45$ b) $x + 10 = 11$
   c) $5x = 65$ d) $5x + 15 = 60$

4. Write an equation in which two inverses will be used to find the solution set.

HOMEWORK:

1. Solve:

   a) $10x - 17 = -7$ b) $24 = 2 + 11x$

   c) $4x - 5 = -9$ d) $-16 = 62 + 13y$

   e) $-41 = 12x - 5$ f) $7 + \frac{1}{4}x = 15$

   g) $\frac{3}{8}x - 1 = 5$ h) $3 + 8x = -29$

2. Find the solution set and check:

   a) $6 - 9x = -39$

   b) $24 = 2 - 11y$

3. Seven more than 3 times a number is 31. Find the number.

4. The width of a singles tennis court is 27 feet. This is one foot more than $\frac{1}{3}$ of the length. What is the length of the court?

5. Mary's age is 3 years less than two times Jim's age. How old is Jim if Mary is 21 years old?

6. Compute the product and check: $(2r - 3s)(-6rs)$

7. Compute the quotient and check.

$$\frac{-24uv^2 + 12u^2v}{-6uv}$$

8. Solve: $|2x + 1| = 9$

Lesson 30 - Solving Eq. Containing Like Terms

AIM: How do we solve equations which contain like terms?

PERFORMANCE OBJECTIVES: The students will be able to...

1. Identify like terms.
2. Recall the procedure for combining like terms.
3. Solve equations which contain like terms on one side.
4. Check the solution.

VOCABULARY: distributive property, like terms

MOTIVATION:

$6x - 2 = 20$ $\qquad$ $6x - 2x = 20$

How are these equations similar? How are they different?

DEVELOPMENT (Include pivotal questions):

1. Discuss motivation. Students may observe:
   a) The same numbers appear in both equations.
   b) The same variable occurs in both equations.
   c) The equation on the right contains two terms involving x.
   d) The procedure learned yesterday may be applied to the equation on the left to obtain the solution $x = 3\frac{2}{3}$.
2. Consider the equation $6x - 2x = 20$. How do we solve $6x - 2x = 20$?
   Refer to the motivation problem. How can we apply the distributive property to combine like terms to obtain simplification?

$$\begin{aligned} 6x - 2x &= 20 \\ 4x &= 20 \\ (\tfrac{1}{4})4x &= (\tfrac{1}{4})20 \\ x &= 5 \end{aligned}$$

Solve equation of type $ax = b$ by using multiplicative inverse.

Check:

$$\begin{aligned} 6x - 2x &= 20 \\ 6(5) - 2(5) &\overset{?}{=} 20 \\ 30 - 10 &\overset{?}{=} 20 \\ 20 &= 20 \checkmark \end{aligned}$$

Ask the students: How do we solve an equation in which like terms appear on one side of the equation?

Elicit:

a) Use the distributive property of multiplication over addition to combine like terms.

b) Solve the resulting equation by using additive inverse, multiplicative inverse or both.

Solve:

$$\begin{aligned} 7y + y - 1 &= 39 \\ 8y - 1 &= 39 \\ \underline{\quad +1} &\underline{= +1} \\ 8y &= 40 \\ (\tfrac{1}{8})8y &= (\tfrac{1}{8})40 \\ y &= 5 \end{aligned}$$

3. Do application 1.

4. Consider more difficult examples.

Solve and check:

$$\begin{aligned} 2x + 4 + x - 3 &= 13 \\ 3x + 1 &= 13 \\ \underline{\quad -1} &\underline{= -1} \\ 3x &= 12 \\ (\tfrac{1}{3})3x &= (\tfrac{1}{3})12 \\ x &= 4 \end{aligned}$$

Check:

$$\begin{aligned} 2x + 4 + x - 3 &= 13 \\ 2(4) + 4 + (4) - 3 &\overset{?}{=} 13 \\ 8 + 4 + 4 - 3 &\overset{?}{=} 13 \\ 13 &= 13 \checkmark \end{aligned}$$

Solve and check:

$$\begin{aligned}73 &= 7a + 5a - 2a + 3\\ 73 &= 10a + 3\\ \underline{-3} &\underline{= \quad\quad -3}\\ 70 &= 10a\\ (\tfrac{1}{10})70 &= (\tfrac{1}{10})10a\\ 7 &= a\end{aligned}$$

Check:

$$\begin{aligned}73 &= 7a + 5a - 2a + 3\\ 73 &= 7(7) + 5(7) - 2(7) + 3\\ 73 &= 49 + 35 - 14 + 3\\ 73 &= 73 \checkmark\end{aligned}$$

5. Do application 2.

6. Solve verbal problems:

   a) Express in terms of x the perimeter of a square whose side is 3x. Answer: $3x + 3x + 3x + 3x = 12x$

   Find x if the perimeter is 144 in.

   Find the side of the square.

   Let $3x$ = side of square

   $$\begin{aligned}3x + 3x + 3x + 3x &= 144\\ 12x &= 144\\ (\tfrac{1}{12})12x &= (\tfrac{1}{12})144\\ x &= 12\end{aligned}$$

   $3(x) = 3(12) = 36$ (side of square)

   b) Express in terms of x the perimeter of a rectangle whose length is $3x + 1$ and whose width is $2x$. Find x if the perimeter is 42.

   Let $2x$ = width, $3x + 1$ = length

   $$\begin{aligned}3x + 1 + 3x + 1 + 2x + 2x &= \text{perimeter}\\ 3x + 1 + 3x + 1 + 2x + 2x &= 42\\ 10x + 2 &= 42\\ \underline{-2} &\underline{= -2}\\ 10x \quad &= 40\\ (\tfrac{1}{10})10x &= (\tfrac{1}{10})40\\ x &= 4\end{aligned}$$

7. Do application 3.

APPLICATIONS:

1. Solve and check:

a)	$7x - 2x = 30$	b)	$3a - a + 4a = 18$
c)	$7y + y = -24$	d)	$-27 = 7c + 3c - c$

2. Solve:
   - a) $8x - 5x + 7 = 19$
   - b) $8x + 4 - 3x = 24$
   - c) $12 = 3c - 12 - 5c$
   - d) $7 = 3a + 5a - a + 7$
3. a) The length of a rectangle is 1 foot longer than three times its width. Find the length and width if the perimeter is 82.
   b) Dad's age is 7 years less than 5 times Junior's age. The sum of their ages is 53. Find each of their ages.

SUMMARY:

1. I am thinking of a number. When I double it and add the original number, I get 27. What is the original number?
2. Why is it important that we check our answer to a problem with the facts of the original problem?

HOMEWORK:

1. Solve and check:
   - a) $4x - x = 21$
   - b) $25 = 3x + 2x$
   - c) $x + 9 + x = 67$
   - d) $8x + 4 + x = 49$
   - e) $y + 5y - 7 = 113$
   - f) $c + c - 13 = 75$
   - g) $18 = 8a - 7 - 3a$
   - h) $32 = 8y + 4 - 4y$
   - i) $-7 = 3a + 5a -a + 7$
2. Zorinda weighs 7 pounds more than Carol. Their combined weight is 203 pounds. How much does each girl weigh?
3. The second of two numbers is 7 times the first. Their sum is 32. Find the numbers.
4. The sum of three numbers is 79. The second number is 9 times the first and the third is 3 more than the second. Find the numbers.
5. The length of a rectangle is 7 centimeters greater than the width. The perimeter is 30 centimeters. Find the length and width.
6. Compute the quotient and check:

$$\frac{-63x^2y + 21y^2}{-3y}$$

7. If y is an integer, solve:

$$(|2 - y| \neq 3) \wedge (|y| < 5)$$

Lesson 31 - Solving Equations which Contain Variables on Both Sides

PERFORMANCE OBJECTIVES: The students will be able to...

1. State the need for removal of the variable from one side.
2. Use previously learned procedures for isolating the variable.
3. Check the results.

VOCABULARY: variable, left member of an equation, right member of an equation

MOTIVATION:

Do these equations have the same solution set? Explain.

a) $3x + 2x = 20$ b) $3x = 2x + 20$

DEVELOPMENT (Include pivotal questions):

1. Refer to the motivation. Review the procedure for solving equation (a). The solution $x = 4$ does not satisfy equation (b).
   Have students notice that the variable x appears in both the left and right members of equation (b).
2. Have students recall the use of additive and multiplicative inverses in transforming an equation such as $ax + b = c$ into simpler equivalent equations.
   Would these principles be helpful in obtaining a simpler equation equivalent to $3x = 2x + 20$ so that the variable appears in one member only?

$$\begin{array}{rcll} 3x &=& 2x + 20 & \\ \underline{-2x} & \underline{=} & \underline{-2x \qquad} & \text{(Additive inverse)} \\ x &=& +20 & \text{(Solution set = } \{20\}\text{)} \end{array}$$

Therefore, 20 is the root of the equation $3x = 2x + 20$. Ask the students: How do we solve an equation which contains the variable on both sides of the equation?

Elicit:

a) Write an equivalent equation in which the variable appears only in one member.

b) Solve the resulting equation.

3. Do application 1.

4. Consider: $8x + 4 = 25 + x$

Ask: On which side shall we collect the terms involving x? (Students will probably suggest the left side.)

Ask: What should we add to each side of the equation to eliminate the variable term from the right side?

Proceed to solve:

$$\begin{array}{rcll} 8x + 4 &=& 25 + x & \\ -x &=& -x & \text{(Additive inverse)} \\ \hline 7x + 4 &=& 25 & \\ -4 &=& -4 & \\ \hline 7x &=& 21 & \\ (\frac{1}{7})7x &=& (\frac{1}{7})21 & \text{(Multiplicative inverse)} \\ x &=& 3 & \end{array}$$

Check:

$$\begin{array}{rcl} 8x + 4 &=& 25 + x \\ 8(3) + 4 &\overset{?}{=}& 25 + 3 \\ 24 + 4 &\overset{?}{=}& 28 \\ 28 &=& 28 \checkmark \end{array}$$

Ask: Could we have added $-8x$ to each side of the equation as the first step in obtaining an equivalent equation with the variable in only one member?

Consider:

$$\begin{array}{rcl} 8x + 4 &=& 25 + x \\ -8x &=& -8x \\ \hline +4 &=& 25 - 7x \\ -25 &=& -25 \\ \hline -21 &=& -7x \\ (-\frac{1}{7})(-21) &=& (-\frac{1}{7})(-7x) \\ +3 &=& x \end{array}$$

Note: The result is the same, whether the terms involving the variable are collected on the left side or the right side.

Solve for x:

$$\begin{array}{rcll}
10 - 6x + 9x & = & x + 2 & \\
10 + 3x & = & x + 2 & \text{(Combine like terms.)} \\
-x & = & -x & \text{(Additive inverse)} \\
\hline
10 + 2x & = & 2 & \\
-10 & = & -10 & \text{(Additive inverse)} \\
\hline
2x & = & -8 & \\
(\frac{1}{2})\,2x & = & (\frac{1}{2})(-8) & \text{(Multiplicative inverse)} \\
x & = & -4 &
\end{array}$$

5. Do applications 2 and 3.

APPLICATIONS:

1. Solve for x.
   a) $8x = 6x + 6$
   b) $-19x = -8x + 22$
2. Solve and check:
   a) $13 + 2x = 3 + 6x$　　b) $7x - 21 = 5x - 9$
   c) $4x - 4 = 3x - 2$　　d) $4x - 5x + 7 = 17 + 9x$
   e) $3z - 2z = -8 - z$　　f) $2y + 0.5 = 2.5 + y$
3. Eight times a number is the same as 90 decreased by the number. Find the number.

SUMMARY:

1. What are the steps to be followed in solving an equation which contains variables on both sides?
2. Solve for x: $ax + b = cx$

HOMEWORK:

1. Solve and check:
   a) $6c - 20 = 2c$　　b) $8y + 11 = 7y$
   c) $7x + 8 = 4 + 3x$　　d) $-x + 3x = 6 + x$
   e) $5x - 2x = 24 - 9x$　　f) $3z + 9 = 1 - z$
   g) $5d + 11 = 8d - 7$　　h) $7x - 0.05 = 3.15 + 2x$
   i) $x + 12 + x = -14$　　j) $8y + 6y - 8 = -y - 38$

2. The sum of twice a number and 6 is the same as 11 increased by the number. Find the number.

3. Sixteen diminished by 8 times a number is the same as 5 diminished by 9 times the number. Find the number.

4. Compute the quotient and check:

$$\frac{-5r^2s^3 + 30r^2s}{.1r^2s}$$

5. Solve: $|7 - 2x| = 5$

*6. Solve for x:

a) $ax + b = cx + d$ b) $ax + b = c + dx$ c) $ax + bx = c + d$

Lesson 32 - Solving Equations Containing Parentheses

AIM: How do we solve equations which contain parentheses?

PERFORMANCE OBJECTIVES: The students will be able to...

1. Identify the parentheses as a grouping symbol.
2. Realize the need for removal of the parentheses.
3. Solve equations containing parentheses preceded by a negative sign, a positive sign or a signed number.
4. Check the solution.

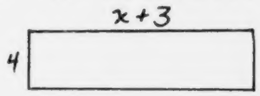
VOCABULARY: distributive property

MOTIVATION:

The length of a rectangle is x + 3 and its width is 4. If the area is 32, find the length.

DEVELOPMENT (Include pivotal questions):

1. Refer to the motivation:



Area = $(x + 3)4$

Let $x + 3$ = length. Area = length x width.

$$\begin{aligned} 32 &= (x+3)4 \\ 32 &= 4x + 12 \\ \underline{-12} &\underline{= \quad\; -12} \\ 20 &= 4x \\ (\tfrac{1}{4})20 &= (\tfrac{1}{4})4x \\ 5 &= x \end{aligned}$$

Remove parentheses using distributive law.

Length = $x + 3 = 5 + 3 = 8$

Check solution by returning to the original problem.

Area = length x width

$32 = 8 \times 4 = 32$

2. Review, using the distributive property of multiplication over addition to simplify each of the following.

		Answers:
a)	$(x + 3)4$	$4x + 12$
b)	$5(x + 2)$	$5x + 10$
c)	$8(2c - 1)$	$16c - 8$
d)	$(x - 5)(-2)$	$-2x + 10$
e)	$3(a - 5)$	$3a - 15$
f)	$2(2a + 1)$	$-4a - 2$
g)	$-2(6 - 9x)$	$-12 + 18x$

3. Solve an equation containing parentheses preceded by a negative signed number.

Consider the equation:

$$\begin{array}{rcl} -5(x + 2) & = & 30 \qquad \text{(Use distributive property)} \\ -5x - 10 & = & 30 \\ \underline{\quad +10} & \underline{=} & \underline{+10} \\ -5x & = & 40 \\ (-\frac{1}{5})(-5x) & = & (-\frac{1}{5})40 \\ x & = & -8 \end{array}$$

Check:

$$\begin{array}{rcl} -5(x + 2) & = & 30 \\ -5(-8 + 2) & \overset{?}{=} & 30 \\ -5(-6) & \overset{?}{=} & 30 \\ 30 & = & 30 \checkmark \end{array}$$

4. Ask the students: How do we solve equations which contain parentheses?

Elicit:

a) Use the distributive property to remove the parentheses.

b) Solve the resulting equation.

5. Solve an equation containing parentheses preceded by a negative sign. Note: This may be understood as multiplying by -1.

Solve:

$$\begin{array}{rcl} -(y + 5) & = & 8 \\ -y - 5 & = & 8 \\ \underline{\quad +5} & \underline{=} & \underline{+5} \\ -y & = & 13 \\ (-1)(-y) & = & (-1)13 \\ y & = & -13 \end{array}$$

Check:

$$\begin{array}{rcl} -(y + 5) & = & 8 \\ -(-13 + 5) & = & 8 \\ -(-8) & = & 8 \\ 8 & = & 8 \end{array}$$

6. Solve:

$$\begin{array}{rcl} 8(2c - 1) & = & 40 \\ 16c - 8 & = & 40 \\ +8 & = & +8 \\ \hline 16c & = & 48 \\ (\frac{1}{16})16c & = & (\frac{1}{16})48 \\ c & = & 3 \end{array}$$

Check:

$$\begin{array}{rcl} 8(2c - 1) & = & 40 \\ 8(2(3) - 1) & = & 40 \\ 8(6 - 1) & = & 40 \\ 8(5) & = & 40 \\ 40 & = & 40 \end{array}$$

7. Do application 1.

8. Consider more difficult equations.
Elicit that in any equation where parentheses appear, the first property to be applied is the distributive property.

Consider:

$$\begin{array}{rcll} 2(x + 3) + 7 & = & 16 & \\ 2x + 6 + 7 & = & 16 & \text{(Combine like terms.)} \\ 2x + 13 & = & 16 & \\ -13 & = & -13 & \\ \hline 2x & = & 3 & \\ (\frac{1}{2})2x & = & (\frac{1}{2})3 & \\ x & = & \frac{3}{2} \text{ or } 1\frac{1}{2} & \end{array}$$

Consider:

$$\begin{array}{rcll} 6x - 3(x - 5) & = & 18 & \\ 6x - 3x + 15 & = & 18 & \text{(Use distributive property.)} \\ 3x + 15 & = & 18 & \text{(Combine like terms.)} \\ -15 & = & -15 & \\ \hline 3x & = & 3 & \\ (\frac{1}{3})3x & = & (\frac{1}{3})3 & \\ x & = & 1 & \end{array}$$

Consider:

$$\begin{array}{rcl} 2(3x - 5) & = & 2(x + 3) \\ 6x - 10 & = & 2x + 6 \\ +10 & = & +10 \\ \hline 6x & = & 2x + 16 \\ -2x & = & -2x \\ \hline 4x & = & 16 \\ (\frac{1}{4})4x & = & (\frac{1}{4})16 \\ x & = & 4 \end{array}$$

9. Do applications 2 and 3.

APPLICATIONS:

1. Solve and check:

   a) $-3(y + 5) = 3$ b) $5(2x - 3) = 15$

2. Solve:
   - a) $2(x + 1) + x = 8$ b) $3(x - 2) + x = 14$
   - c) $7y - 2(8y + 1) = 4$ d) $4x - (x + 2) = 13$
   - e) $7(2 + 2x) = 4(-x - 10)$

3. The length of a rectangle is $2x - 3$ and its width is 5. Find the length if the area is 35.

SUMMARY:

1. Explain why $4(x + 3) = 32$ and $4x + 3 = 32$ are not equivalent equations.

2. What are the steps to be followed in solving an equation containing parentheses?
   - a) Remove parentheses.
   - b) Combine like terms.
   - c) Apply additive inverse.
   - d) Apply multiplicative inverse.
   - e) Write the solution set.
   - f) Check in the original equation.

3. Write an equation containing parentheses whose solution set is $\{0\}$.

4. Write an equation containing parentheses which has no solution.

HOMEWORK:

1. Solve and check:
   - a) $4(x - 2) = 20$ b) $2(2s + 3) = -18$
   - c) $4(2x - 1) = 20$ d) $3(x + 4) - x = 18$

2. Solve:
   - a) $15 - (5x - 4) = 34$ b) $8x = 2(x + 15)$
   - c) $2(3x - 5) = 2x + 6$ d) $10(x + 2) = -2(6 - 9x)$
   - e) $8y - 2(-3y + 4) = -(y + 38)$

3. Five times the sum of a number and 2 is 25. Find the number.

4. Seven times the sum of a number and 4 is the same as 8 decreased by 3 times the number. Find the number.

5. Six times the sum of a number and -1 is the same as 2 more than 8 times the number.

6. If the area of a rectangle is $24r - 6r^2s$ and one side is $2r$, compute the perimeter.

*7. Solve: $36 - 4x = -10$

Lesson 33 - Inequalities

AIM: How do we solve an inequality?

PERFORMANCE OBJECTIVES: The students will be able to...

1. Recall the meaning of an inequality.
2. State the differences in the procedures for solving an equation and an inequality (i.e., division or multiplication by a negative number reverses the sense of the inequality).
3. Solve first degree inequalities.
4. Graph the solution on a number line.
5. Check a member of the solution.

VOCABULARY: inequality

MOTIVATION:

Insert the appropriate comparison symbol (< or >) between each pair of numerals:

a) $10 \quad 6$ b) $10 + 2 \quad 6 + 2$

c) $10 - 2 \quad 6 - 2$ d) $2(10) \quad 2(6)$

e) $(-2)(10) \quad (-2)(6)$ f) $\frac{10}{2} \quad \frac{6}{2}$

g) $\frac{10}{-2} \quad \frac{6}{-2}$

What patterns do you observe here?

DEVELOPMENT (Include pivotal questions):

1. Students should observe that they are comparing the results of performing the same operation on 10 and 6 in each case. The sense of the inequality is preserved in all cases except e and g; when the numbers are multiplied or divided by the same negative quantity, the sense of the inequality is reversed (a number line may be used to illustrate this).

Pivotal questions: What happens when equal quantities are each multiplied by the same negative value? (The results are equal.) Can we expect the rules used for solving equations to apply to inequalities also? (Not entirely.) What cases will require special care? (Multiplication or division by a negative value reverses the sense of the inequality.) It may be desirable to reinforce these points by examining other numerical illustrations.

2. State the following rules for operating on inequalities:
   a) If the same value (positive or negative) is added to or subtracted from each member, the sense of the inequality is preserved.
   b) If each member is multiplied or divided by the same positive value, the sense of the inequality is preserved.
   c) If each member is multiplied or divided by the same negative value, the sense of the inequality is reversed.

   Note: These rules are identical with the rules for operating on equations, except for case c.

3. Elicit the meanings of the comparison symbols $\leq$ and $\geq$ for compound inequalities:

   $x \leq y$ is read, "x is less than or equal to y."
   $x \geq y$ is read, "x is greater than or equal to y."

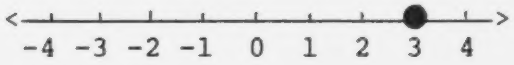
4. How many elements are in the solution set of $x < 3$? (Infinitely many.)
   How do we indicate this solution set on a number line?
   How can we distinguish the solution set for $x < 3$ and the solution set for $x \leq 3$?
   Illustrate the number line graphs for $x = 2$, $x > 2$, $x < 2$, $x \geq 2$, $x \leq 2$, $2 < x$, $2 > x$.

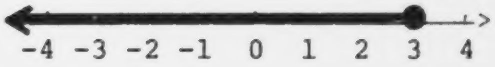
5. Consider: $2x + 5 = 11$
   $2x + 5 \leq 11$

   Find the solution set for each and graph on the number line.

$$\begin{aligned} 2x + 5 &= 11 \\ \underline{\quad -5} &\underline{= -5} \\ 2x &= 6 \\ (\tfrac{1}{2})(2x) &= (\tfrac{1}{2})(6) \\ x &= 3 \end{aligned}$$

$$\begin{aligned} 2x + 5 &\le 11 \\ \underline{\quad -5} &\underline{= -5} \\ 2x &\le 6 \\ (\tfrac{1}{2})(2x) &\le (\tfrac{1}{2})(6) \\ x &\le 3 \end{aligned}$$



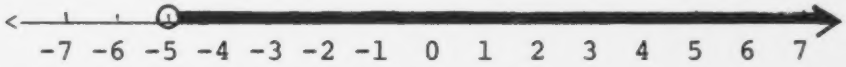


How would the solution set and graph change if the inequality were changed to $2x + 5 < 11$? $2x + 5 > 11$? $2x + 5 \ge 11$?

6. Consider:

$$\begin{aligned} -3x + 7 &< 22 \\ \underline{\quad -7} &\underline{= -7} \\ -3x &< +15 \\ (-\tfrac{1}{3})(-3x) &> (-\tfrac{1}{3})(+15) \\ x &> -5 \end{aligned}$$

(Inequality reverses when each side is multiplied by a negative.)



7. Have the class solve, graph the solution set on a number line, and check one member of the solution set of each of the following.
   a) $3x > 8$
   b) $3x < -10 - 2x$
   c) $-4 \ge 4x + 8$
   d) $-2x > 6$

APPLICATIONS:

For each of the following, solve, graph the solution set on a number line and check an element of the solution set.

a) $-3x \ge 15$

b) $-6y < -12$

c) $7x + 3 \le -25$

d) $5a - 4 < 21$

e) $1 - 2x > 7$

f) $6x - 5 \ge 10x + 15$

SUMMARY:

1. What are the key differences between the techniques for solving an equation and an inequality?

2. What are the differences in their solution sets?

HOMEWORK:

1. Solve each of the following inequalities and graph the solution set.

   a) $2x \geq -32$ b) $y + 3 < 2$
   c) $3z - 1 < -4$ d) $-3u + 1 \leq -5$
   e) $-4v - 1 \geq 3$ f) $2x + 1 \neq 6$

2. Solve each of the following assuming that the domain of each variable is the set of real numbers, and graph the solution set.

   a) $-3x < 2$ b) $8 - 3y \geq 2$
   c) $2(x - 1) = 3(x + 2)$ d) $2(x - 1) < 3(x + 2)$
   e) $2(x - 1) > 3(x + 2)$ f) $x - 3 \leq 5$

*3. Solve each of the following inequalities, assuming that the domain of each variable is an integer between -5 and +5 inclusive, and graph the solution set.

   a) $2x > -4$ b) $3y \leq -12$
   c) $-r > -1$ d) $-2s \geq 6$
   e) $5 + t > 0$ f) $-5 + u < 3$

4. The perimeter of a rectangle is 72 meters. If the length is 6 meters more than 3 times the width, what is the length of this rectangle?

5. Compute the quotient and check your answer:

$$\frac{-48r^2st^3 + 27rst}{-3rt}$$

Lesson 34 - Literal Equations

AIM: How do we solve a literal equation?

PERFORMANCE OBJECTIVES: The students will be able to...

1. Define a literal equation.
2. Solve a literal equation for an indicated variable.

VOCABULARY: literal equation

MOTIVATION:

Solve each of the following equations for x:

a) $3x + 7 = 19$ b) $3x + b = c$

DEVELOPMENT (Include pivotal questions):

1. Discuss solutions to motivation problems:

   a) $x = 4$ b) $x = \frac{c-b}{3}$

   Note that the first equation has a specific, numerical solution, while the solution to the second equation is expressed in terms of other variables.

   Pivotal question: What similarities and differences can be observed in the two equations in the motivation exercise?

   Difference: Equation b involves several variables.

   Define: A literal equation is an equation containing more than one variable or letter.

   Similarity: Equations a and b are similar in structure and method of solution.

   Elicit that literal equations are solved by the same techniques used for equations with numerical coefficients.

   A useful strategy for dealing with complicated literal equations is to form a similar equation with numerical coefficients, solve it and then proceed in an analogous fashion with the given literal equation.

2. Solve each of the following for the variable indicated in parentheses:

   a) $ax = b$ (x) b) $ax + b = c$ (x)
   c) $ax - by = c$ (x) d) $ax - by = c$ (y)
   e) $\frac{ax}{b} = c$ (x) f) $3(x - 2y) = 5$ (x)
   g) $\frac{2}{3}y = x$ (y) h) $\frac{3}{7}p - xs = w$ (p)

APPLICATIONS:

Be sure to explain the meaning of each of the following formulas as it is presented. Solve each of the following for the variable indicated in parentheses.

a) $D = RT$ (R) b) $I = PRT$ (T)
c) $A = \frac{1}{2}bh$ (b) d) $P = 21 + 2w$ (w)
e) $F = \frac{9}{5}c + 32$ (c) f) $A = \frac{h(b + c)}{2}$ (c)

SUMMARY:

1. What is a literal equation?
2. How do we solve a literal equation for an indicated variable?
3. In what practical situations would you need to use a literal equation?

HOMEWORK:

1. Solve each of the following for the variable indicated in parentheses:

   a) $50 = 2h + 30$ (h) b) $P = 2k + 30$ (k)
   c) $P = 2k + 2w$ (k) d) $48 = \frac{8(b + y)}{2}$ (y)
   e) $A = \frac{h(b + 4)}{2}$ (b) f) $A = \frac{h(b + c)}{2}$ (h)

2. Solve and plot the solution set:

   a) $-3x \leq x + 12$ b) $2(y - 3) \geq 10$
   c) $3(5 - z) < -12$ d) $2x - 3 \leq 9$

3. The sum of Mary's and John's ages is 30 years. If Mary's age is 6 years less than twice John's age, how old are they?

4. Compute the quotient and check:

$$\frac{-48rs^2 - 18r^2s}{6r^{-1}s}$$

Lesson 35 - Number Problems

AIM: How do we solve a number problem?

PERFORMANCE OBJECTIVES: The students will be able to...

1. Write a representation for the variable(s).
2. Write an equation which can be used to solve the problem.
3. Solve the equation.
4. Substitute the solution in the representation.
5. Check the solution in the number problem.

VOCABULARY: separate

MOTIVATION:

I am thinking of a number. If 5 times my number is increased by 8, the result is 4 less than 3 times the number. What number do I have in mind?

DEVELOPMENT (Include pivotal questions):

1. Discuss motivation. Answers that students suggest should be checked in the given sentence. (The correct answer is -6.)
2. Elicit that the translation of verbal to algebraic statements and the methods for solving linear equations may be used to solve word problems.
3. Have the motivation problem translated and solved. ($5x + 8 = 3x - 4$) During the solution, be sure to emphasize the following:
   a) Since the variable is being introduced, from the wording of the problem, it must be explained (quite specifically) what the variable represents. This is often called "the legend" (let x = the numbber the teacher has in mind).

   b) The answer to the problem should be written out in a sentence. (i.e., "The number is -6" is preferable to just $x = -6$.)
   c) The "check" must be done by using the original wording of the problem rather than a derived equation which may contain a translation error.

4. List the 5 basic steps used in the solution of any verbal problem.
   a) Represent the variable(s).
   b) Write the equation(s).
   c) Solve the equation(s).
   d) Write the answer sentence.
   e) Check.

5. Consider: The larger of two numbers is five more than three times the smaller. The difference between the numbers is 13. Find both numbers.
   Elicit that expressions are needed to represent both numbers.
   Legend: x = the smaller number
   $3x + 5$ = the larger number
   Equation: $3x + 5 - x = 13$
   Answer: The larger number is 17 and the smaller is 4.

6. Consider: Separate 30 into two parts such that twice the larger equals 3 times the smaller.
   It may not be obvious to the class that the numbers can be represented by the expressions x and $30 - x$. to elicit this idea, suggest that specific values be considered and construct a table such as:

larger	smaller
20	(30 - 20 = 10)
25	(30 - 25 = 5)
21	(30 - 21 = 9)
x	$30 - x$

   Note: In this problem, x could be used to represent either the larger or the smaller number. The teacher may wish to ask: Suppose x were used to represent the smaller number; how would the larger number be represented? $(30 - x)$ If time permits, both solutions may be completed and compared.

Legend: x = the larger number

$30 - x$ = the smaller number

Equation: $2x = 3(30 - x)$

Answer: The larger part is 18 and the smaller is 12.

APPLICATIONS:

1. If 3 times a number is decreased by 2, the result is 34. Find the number.
2. If twice a number is increased by 7, the result is the same as if 5 times the number is diminished by 2. Find the number.
3. If 8 times a number is subtracted from 57, the result is 11 times the number. Find the number.
4. One number is 3 more than another number. If the sum of the numbers is 19, find both numbers.
5. One number is 4 less than twice another number. If the sum of the numbers is 80, find both numbers.
6. Separate 45 into two parts such that one part is 6 more than 2 times the other.

SUMMARY:

1. What procedure should be used in solving verbal problems?
2. What operation is implied in the direction "separate"?

HOMEWORK:

1. If three times a number is decreased by 5 and the result is doubled, the final result is 74. Find the original number.
2. If two times a number is increased by 17, the result is equal to four times the number. Find the number.
3. Separate 32 into two parts such that the larger is 6 more than the smaller.

4. George had \$45. He spent twice as much as he saved. How much did he spend?

5. The longest side of a triangle is twice the shortest side and 7 cm longer than the third side. If the perimeter is 47 cm, how long is each side?

6. The length of a rectangle is 3 inches less than twice the width. If the length is 48 feet, what is the perimeter of the rectangle?

7. Write a number problem for which the answer is 7.

8. Solve for the variable in parentheses:

   a) $\frac{a+b}{2} = v$ (a) b) $\frac{a+b+c}{3} = v$ (a)

   c) $\frac{a+b}{d} = v$ (a) d) $|5 - 3x| < 44$ (x)

9. Express as a fraction with positive exponents:

   $4a^2b^{-2}c^{-1}$

Lesson 36 - Consecutive Integer Problems I

AIM: How do we solve a consecutive integer problem?

PERFORMANCE OBJECTIVES: The students will be able to...

1. Give examples of:
    a) consecutive integers
    b) consecutive even integers
    c) consecutive odd integers
2. State representations for the above.
3. Write an equation for, solve and check consecutive integer problems.

VOCABULARY: integer, even integer, odd integer, consecutive

MOTIVATION:

1. Last April, it rained on five consecutive days. It started raining on the 12th, what was the date of the last of these rainy days?
2. Bill is looking for his friend's house. He stops at number 42 Euclid Ave. to ask directions, and is told to go to the third house up on the same side of the street. If the numbers increase consecutively and evenly as he continues on his way, what is his friend's address?

DEVELOPMENT (Include pivotal questions):

1. Discuss motivation. In each case, list the sequence of numbers involved:

    Rain on five consecutive days: 12,13,14,15,16
    Last rainy day - April 16
    House numbers on this side of the street are even
        42,44,46,48
    The third house from #42 is 48 Euclid Ave.

2. Elicit the definition of integer, consecutive integers, even integer, consecutive even integers, odd integer, consecutive odd integers.

3. Elicit numerical examples to illustrate the definitions:
   Name four consecutive integers beginning with 17.
   Name five consecutive integers beginning with 43.
   Name seven consecutive integers beginning with -4.

   Is -4 odd or even?

   Note: Zero is an even integer.

4. Students should observe that consecutive integers are separated by 1, while consecutive even integers and consecutive odd integers are separated by 2.

   Pivotal questions:
   a) If n represents an integer, how would we represent the next 3 consecutive integers?
   b) If x represents an even integer, how would we represent the next three consecutive even integers?
   c) If y represents an odd integer, how would we represent the next three consecutive odd integers?

   Students should conclude that the following algebraic representations may be used in the general case.
      1. Consecutive integers: x,x+1,x+2,x+3,x+4...
      2. Consecutive odd or even integers:
         x,x+2,x+4,x+6,x+8...

   Note: Explain why x, x + 1, x + 3, x + 5 is incorrect as a representation of consecutive odd integers.

APPLICATIONS:

Reemphasize the five basic steps which will be utilized to solve verbal problems, and solve:

1. Find three consecutive integers whose sum is 45.
2. Find three consecutive integers whose sum is 63.
3. Find four consecutive even integers whose sum is 36.
4. The sum of the first and last of four consecutive integers is 33. Find the sum of all four integers.

SUMMARY:

1. Give an example of
   - a) 4 consecutive odd integers
   - b) 4 consecutive integers
   - c) 4 consecutive even integers
2. Why are consecutive even and odd integers represented in the same way?
3. Can you find three consecutive even integers whose sum is 53? Explain.

HOMEWORK:

1. Solve for the indicated variable:
   - a) $\frac{a + 2b + 3c}{6} = v$ (c)
   - b) $-3(x - 5) < 2(x - 10)$ (x)
   - c) $17 - 2x < 4$ (x)
2. If 3 consecutive integers total 18, what are they?
3. If 3 consecutive even integers total 18, what are they?
4. If the sum of the greatest and the smallest of 4 consecutive odd integers is 30, what is the sum of all four integers?
5. If $a = 2$, $b = -3$, $c = 4$, evaluate:
   - a) bc^2
   - b) $a^{-3}c^2$
   - c) $\frac{a-b}{c}$
   - d) $3a^2 - 4b$
   - e) $a^{-1} + b^{-1}$
   - f) $\frac{1}{a^{-1} + c^{-1}}$

Lesson 37 - Consecutive Integer Problems II

AIM: How do we solve more complex consecutive integer problems?

PERFORMANCE OBJECTIVES: The students will be able to...

1. Give examples of:
    a) consecutive integers
    b) consecutive even integers
    c) consecutive odd integers
2. State representations for the above.
3. Write an equation for, solve and check consecutive integer problems.

VOCABULARY: no new terms

MOTIVATION:

1. Represent six consecutive odd integers starting with a.
2. If 3x - 5 represents an even integer,
    a) represent the next three consecutive even integers.
    b) represent the sum of these four even integers.
    c) what conclusions can be drawn about x?

DEVELOPMENT (Include pivotal questions):

1. Discuss motivation. Note: x must be an odd integer.
2. Since this lesson extends material in Lesson 36, the teacher may wish to review the part of the homework involving consecutive integer problems before proceeding to the more challenging questions which follow.
3. Find 4 consecutive even integers such that the sum of the smallest and 3 times the largest is 106.
4. Find 3 consecutive integers such that the largest is 16 less than the sum of the other two.

APPLICATIONS:

1. Find 3 consecutive odd integers such that the sum of four times the first and three times the second is equal to five times the third.
2. If (2n + 1) represents an odd integer,
   a) what are the next 3 consecutive odd integers?
   b) what are the preceding 3 consecutive even integers?
3. If (3x - 5) represents an odd integer,
   a) what are the next 3 consecutive even integers?
   b) what are the preceding 3 consecutive even integers?
4. If 3 times the first of 3 consecutive integers is subtracted from twice the second, the result is 16 more than the third. Find the integers. (More challenging: answers are negative integers.)

SUMMARY:

1. What is the meaning of "consecutive"?
2. Identify situations in life that involve consecutive integers, consecutive even integers or consecutive odd integers.

HOMEWORK:

1. If n represents the middle of three consecutive integers, represent the other two.
2. If n represents the middle of three consecutive even integers, represent the other two.
3. How is the middle of three consecutive integers related to their sum?
4. Find three consecutive integers such that the sum of the smaller two equals the third.
5. Find four consecutive even integers such that the sum of the first and the third is six less than the largest.
6. Find four consecutive odd integers such that the largest equals two more than the sum of the other three.

7. Find three consecutive odd integers such that four times the sum of the first and second exceeds the largest by 53.

8. Solve:

   a) $\frac{a + 2b + 3c}{d} = v$ for c.

   b) $8 + 2x \leq 6$

Lesson 38 - Motion Problems I

AIM: How do we solve a motion problem?

PERFORMANCE OBJECTIVES: The students will be able to...

1. Distinguish between different types of motion problems (opposite direction vs. same direction).
2. State the basic formula: Rate x Time = Distance.
3. Solve and check motion problems of the type where the objects are traveling in opposite directions.
4. Write the answer so that it answers the question in the problem.

VOCABULARY: rate, time, distance, m.p.h.

MOTIVATION:

Suppose your family plans a trip to visit your aunt in Philadelphia. What time should you leave to arrive in time for a 2 PM dinner?
What additional information would you need in order to answer this question?

DEVELOPMENT (Include pivotal questions):

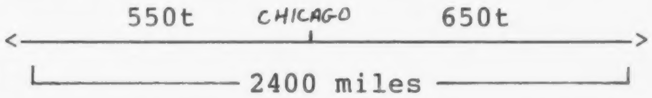
1. A discussion of the motivation exercise should produce the following results:
   a) there are three factors involved in any motion problem: distance; time and rate (typical units should be included).
   b) a formula connects these factors: D = RT
2. A table listing rates, times and distances may be used to assist in the solution of the following exercise:
   How far can a plane travel if it averages 600 m.p.h. for
   1) 5 hrs. 2) 8 hrs. 3) x hrs. 4) (x + 4)hrs.?

3. Consider: Two planes leave Chicago at the same time and travel in opposite directions. If one plane flies at the rate of 550 m.p.h. and the other at a rate of 650 m.p.h., in how many hours will they be 2400 miles apart?

Pivotal questions:

a) What are we asked to find in this problem? (Write Legend: t = number of hours each plane travels until they are 2400 miles apart.)

b) What can we do to help us picture what it's happening in this problem?
(Elicit strategy: make a diagram.)



c) What is the formula that connects the quantities, rate, time and distance?

d) What equation connects the distances in our diagram? Why?

e) What is the answer? (Insist on a written sentence.)

APPLICATIONS:

1. Shelly and Esther start out at 10AM at points which are 35 miles apart and travel toward each other. Shelly walks at the rate of 4 m.p.h., while Esther rides her bicycle at the rate of 10 m.p.h. At what time will they meet? (Note: Hours elapsed must be converted to a time of day.)

2. Two cars leave from the same point traveling in opposite directions with one car averaging 4 m.p.h. more than the other. After 2 hours, the cars are 212 miles apart. find the rate of each car.

SUMMARY:

1. What are the factors which must be considered when solving motion problems?

2. What is the relationship which connects these factors?

3. If you make a second trip over the same distance at a slower rate, how will the time needed for the second trip compare to the first?

4. Compare the last three problems done in this lesson. How are they similar? How are they different?

HOMEWORK:

1. Two planes travel in opposite directions at 300 m.p.h. and 500 m.p.h. In how many hours will they be 2,000 miles apart?

2. John and Henry left home at the same time. John walked toward the beach at the rate of 3 m.p.h. Henry jogged in the opposite direction at 7 m.p.h. In how many hours will John and Henry be 15 miles apart?

3. Two ships which are at ports 310 miles apart start sailing towards each other at 11 AM. If their average rates are 22 mph and 34 mph, at what time will they be 30 miles apart?

4. Find four consecutive even integers with the property that the sum of the smallest and the largest equals three times the next-to-smallest.

5. Solve and check.
   a) $2(7 - 3x) = 5(2 - x)$
   b) $7 - 4y = 3(2 - y)$
   c) $5 + 3z \leq 29$

6. Compute the product and express the result with positive exponents:

$$(3x^{-2}yz^{-3})(-4x^{2}y^{-2}z)$$

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Lesson 39 - Motion Problems II

AIM: How do we solve motion problems when one object is to overtake to a second object?

PERFORMANCE OBJECTIVES: The students will be able to...

1. State the difference between these problems and the type from the previous lesson.
2. Solve overtake problems.
3. Write the answer so that it answers the question in the problem.

VOCABULARY: overtake

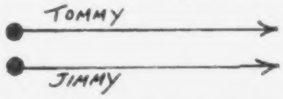
MOTIVATION:

At 1 PM, Tommy left on a trip, driving at 40 mph. Two hours later, his brother Jimmy discovered that Tommy had forgotten his suitcase. Putting it in his own car, he went after hos brother on the same road. At 7 PM, he caught up with his brother. What was Jimmy's average rate?

1. Make a diagram to illustrate this problem.
2. Before solving the problem, estimate the answer.

DEVELOPMENT (Include pivotal questions):

1. Discuss motivation problem.



Students should observe that Jimmy must be travelling faster than Tommy in order to overtake him, because he has less time on the road.

Pivotal question: What component of motion is the same for the two brothers? (distance)

Tommy travels 6 hours at 40 mph: a total of 240 miles. to travel 240 miles in 4 hours, Jimmy must have an average rate of 60 mph.

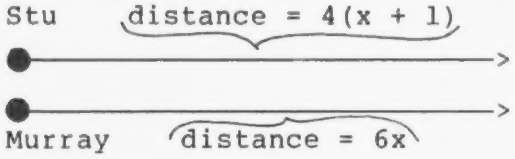
This problem will review the components of motion and the formula learned in lesson 38.

2. Consider the following problem: Stu goes for a walk in the country and travels at the rate of 4 m.p.h. An hour later, Murray leaves from the same place and pedals his bicycle along the same path at the rate of 6 m.p.h. How many hours will it take Murray to overtake Stu?

   Ask: What are we looking for?

   (Write Legend: let x = number of hours Murray travels before overtaking Stu.)

   Ask: How can we represent the situation with a diagram?



$$6x = 4(x + 1)$$
$$x = 2$$

   Answer: Murray overtakes Stu after travelling for two hours.

   (The teacher may wish to use a chart with columns headed rate, time and distance if students have difficulty in expressing distances traveled by each boy.)

   Pivotal question: In overtake problems involving motion over the same path, what quantity will always be the same for both travelers?

   Enrichment: The teacher may wish to explore the solution of this problem with x used to represent Stu's travelling time.

APPLICATIONS:

1. A plane left Kennedy Airport at 5AM flying east at the rate of 550 m.p.h. At 7AM another plane left the same airport flying east at the rate of 825 m.p.h. At what time does the second plane overtake the first?
2. Contrast this problem with the problem above:
   At 5AM, two trains leave the same city and travel in opposite directions. Their rates differ by 5 m.p.h. At

11AM the trains are 450 miles apart.

a) Find the rate of each train.

b) How far has each train traveled?

SUMMARY:

1. What are the special features of "overtake" problems?
2. What strategies are helpful in proceeding from the statement of a motion problem to its solution?

HOMEWORK:

1. A steamship left San Francisco for Hawaii traveling at the rate of 50 m.p.h. Six hours later a plane is sent out to overtake the steamship. If the plane's speed is 300 m.p.h., how long will it take for the plane to catch up with the steamship? How far will they be from San Francisco?
2. Mary left home walking toward a park at the rate of 3 m.p.h. John left an hour later. If he walks at the rate of 4 m.p.h., in how many hours will he catch up to Mary?
3. At 10AM two cars started at the same point and traveled in the same direction. One car averaged 53 m.p.h. while the other averaged 44 m.p.h. In how many hours will the two cars be 36 miles apart?
4. Jack left one end of a lake to row to the opposite end at the same time that Jill left the opposite end to meet Jack. If Jack rowed 3 m.p.h. while Jill rowed 2 m.p.h., how long will it take for them to meet if the distance between them was initially 4 miles?
5. If m is the average of the smallest and largest of four consecutive odd integers, represent all four integers in terms of m.
6. Solve $|x| - x = 6$.

Lesson 40 - Decimal Equations

AIM: How do we solve a decimal equation?

PERFORMANCE OBJECTIVES: The students will be able to...

1. Solve a decimal equation by clearing the decimal.
2. Check the solution for a decimal equation.

VOCABULARY: power of 10

MOTIVATION:

1. Complete this table, multiplying mentally each of the given numbers by 10, 100 and 1,000.

x	35.76	.6	4
10			
100			
1000			

2. a) (.03)(__) = 3   b) (.2)(__) = 2   c) (.005)(__) = 5

DEVELOPMENT (Include pivotal questions):

1. Use the first part of the motivation exercises to review the rules for multiplying by powers of 10. Use part 2 to exhibit that the multiplication of a decimal by a power of ten can eliminate the decimal.
   The class should conclude: To transform a decimal number to an integer, multiply it by a power of ten; the exponent of the power of ten should be equal to the number of decimal places in the original decimal.
2. Consider the equation .03x = 12. How can we solve for x? The discussion of the problem should bring out the following:
   Although it is possible to solve the equation by dividing each side by .03, it is probably easier to transform the equation into an equivalent one with integer coefficients by multiplying each side by 100.

3. Consider: $1.025x + 2.7 = 45.75$

   Pivotal question: What power of ten should we use a the multiplier to obtain an equivalent equation with integer coefficients? Explain why.

   (Complete the solution and check the answer in the original equation.)

APPLICATIONS:

1. Solve each of the following:

a)	$.6a = 54$	b)	$.04y = 20$
c)	$-.12k = 36$	d)	$.02x = 1.2$
e)	$.3b = .27$	f)	$.4x + 2 = 3.4$
g)	$5 - .02y = .48y$	h)	$-8.1k + 2.7k = 81$
i)	$.24x - 3.6 = 54$	j)	$2.5x + 1.24 = x + 3.69$

   Be sure to stress that both sides of the equation must be multiplied by the same power of ten and that the multiplier to be used is determined by the term which has the greatest number of decimal places in its coefficient. Check some of the solutions.

SUMMARY:

1. Explain what happens when a decimal number is multiplied by

   a) 10^5 b) 10^{25} c) 10^{-3} d) 10^0

2. How may we solve an equation which contains decimal coefficients?

HOMEWORK:

1. Solve and check:

a)	$.3x = 6$	f)	$.2(u + 1) = 8$
b)	$3y = .6$	g)	$-.2(v + 1) = 8$
c)	$.3x = .6$	h)	$-.2(w + 1) = 8$
d)	$.2r + .3r = 2$	i)	$.5x = 1 + .6x$
e)	$.2s - .3s = 2$	j)	$.01(y - 2) = .5y$

2. Find five even consecutive integers that have a sum of 100.

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3. Are there five odd consecutive integers that have a sum of 100?

4. A car leaves New York for Albany averaging 50 m.p.h. A motorcycle leaves New York for Albany 90 minutes later and averages 60 m.p.h. How long will it take to catch up to the car?

*5. Compute: $(.2)^{-2}(4)^{-1}(-7)^{0}$

6. Solve: $.1 - x \le 1$

Lesson 41 - Percentage Problems

AIM: How do we solve problems involving percentage?

PERFORMANCE OBJECTIVES: The students will be able to...

1. Represent a percent as a fraction and as a decimal.
2. Solve simple problems using percent (sales tax, discounts, interest, etc.).

VOCABULARY: sales tax, discount, interest

MOTIVATION:

Upon entering a store you find that all items with a red tag can be discounted 25% while items with green tags can be discounted 40%. If we ignore sales tax, how much would each of the following sell for?

- a) a red-tagged portable stereo cassette player marked at $200
- b) a green-tagged leather jacket marked at $300

DEVELOPMENT (Include pivotal questions):

1. The discussion of the above exercises should produce the following results:
   - a) A 25% discount does not mean $25 off the price.
   - b) Original Price - Discount = Sale Price
   - c) Discount can be calculated by multiplying the % discount (written as a decimal or fraction) by the original price.
   - d) Alternate strategies should be encouraged. For example: if you get a 25% discount, the sale price will be 75% of the original price (or $\frac{3}{4}$ of the original price).

2. Review, as needed, that 25% means $\frac{25}{100} = \frac{1}{4} = .25$ and that 40% means $\frac{40}{100} = \frac{2}{5} = .40$. Define percent as hundredth ($x\% = \frac{x}{100}$). Before going to further examples, provide drill exercises on exchanging percents to decimals and fractions:

   Change each of the following to a decimal and to a fraction in lowest terms:

   a) 75% b) 20% c) 60% d) 8% e) $33\frac{1}{3}\%$ f) $12\frac{1}{2}\%$

APPLICATIONS:

1. Find the sales price of each of the following:
   a) A $15 shirt which is discounted 10%.
   b) A $180 watch which is discounted 30%.
   c) A $60 pair of shoes which is discounted 35%.
2. If there is an 8% sales tax on each of the above sales, find the final customer cost in each case.
   a) (optional) Does an 8% discount and an 8% sales tax mean that the item is selling for the original price?
3. A yearly 15% interest fee is charged when money is borrowed at a certain bank. What is the interest on a $500 loan for one year?
4. The same bank pays 12% interest on special savings certificates. How much interest will you receive on a $7000 certificate for one year?

SUMMARY:

1. How do we change a % to an equivalent fraction?
2. How do we change a % to an equivalent decimal?
3. How do we calculate the sale price of an item which is discounted?
4. Write $66\frac{2}{3}\%$ as a decimal.

   Note: Explain that .66 is incorrect because it is too small. Students should be encouraged to memorize the percent equivalents of $\frac{1}{3}$ and $\frac{2}{3}$.

HOMEWORK:

1. Find the sale price of each of the following assuming no sales tax:
   a) a $25 shirt which is discounted 6%
   b) a $1250 motorcycle which is discounted 15%

2. Find the sale price of each of the problems in 1 assuming an 8% sales tax.

3. A bank is giving $9\frac{1}{2}$% interest on money left in a certain savings account. If $800 is left in the bank, how much is your account worth at the end of one year? If you make no withdrawal, how much will be in the account at the end of two years?

4. John owns a rare coin, the value of which has increased 200% since he bought it. If he paid $50 for it, what is its current value?

5. Solve and check:
   a) $.08x = 32$
   b) $.0825y = 49.5$

6. Solve and check: $-3(2u - 1) = 11 - u$

Lesson 42 - Verbal Problems Involving Percent

AIM: How do we solve a verbal problem involving percent?

PERFORMANCE OBJECTIVES: The students will be able to...

1. Represent the condition of the problem in an equation.
2. Solve the equation.
3. Examine the feasibility of the final answers.

VOCABULARY: no new terms

MOTIVATION:

The same electric typewriter is on sale at a discount at two different stores, one of which does not charge sales tax for purchases by schools.

SMITH'S STATIONARY	OSCAR'S OFFICE SUPPLIES
$210	$220
30% off	40% off
no sales tax	8% sales tax

Which looks like the best buy? Which is the best buy? How much do you save?

DEVELOPMENT (Include pivotal questions):

1. Use the motivation exercises to briefly review the meaning of percent and the technique for finding a percent of a number.
   The school can save $4.44 by buying at Oscar's ($147 - $142.56).
2. Referring to the solution just completed, the teacher should write these sentences on the board:
   30% of 210 is 63
   40% of 220 is 88
   and challenge the students to rewrite each sentence as an equation.

30% of 210 is 63 ——> (.30)(210) = 63

40% of 220 is 88 ——> (.40)(220) = 88

Stress that "of" implies multiplication in this context.

3. Pose: If 25% of a number is 12, find the number.

   Ask: How shall we represent this verbal problem as an equation?

   $$.25x = 12$$
   $$x = 48$$

   Check: 25% of 48 is 12. (?)

   $$(.25)(48) \overset{?}{=} 12$$
   $$12 = 12 \checkmark$$

   Note: Some students may think that the answer is 3 (having found 25% of 12). This idea may be debunked by writing "25% of 3 is 12" and asking, "Is this reasonable?" If should be obvious that since 12 > 3, 12 cannot possibly equal $\frac{1}{4}$ of 3.

4. If 4% of a number is 32, find the number.

5. If 80% of the students in a class passed an examination and 32 passed, how many students were in the class?

6. 45 is 125% of a number. What is the number?

APPLICATIONS:

1. In 1984 the Triangle Manufacturing Company made a profit of $360,000. If this represented 12% of the total sales volume, find this volume.

2. On a typical day a merchant finds that he has collected $175 in sales tax. If sales tax is 8%, how much money did he collect from the sale of merchandise?

SUMMARY:

1. Write the equation you would use to find the unknown number:
   a) A number is 45% of 300.
   b) 45% of a number is 300.
   c) 300 is 45% of a number.

2. What does percent mean?

HOMEWORK:

1. Solve and check each of the following problems by specifying the meaning of a variable, translating the problem into an equation, and solving the equation.
   a) 5% of some number is 24. Find the number.
   b) What percent of 12 is 3?
   c) Some number is 30% of 50. What is this number?
2. In a certain first grade class 25% of the children contracted measles. If 7 children got measles, how many did not?
3. In P.S.999 75% of the children passed the minimal fitness examination. If 676 were examined, how many failed the examination?
4. $6\frac{1}{2}\%$ of a number is 26. find the number.
5. If the sum of 5 consecutive odd integers is 5, what must they be?
6. Solve: $.2 + x \leq 2$.

Lesson 43 - Verbal Problems (Inequalities)

AIM: How do we solve a verbal problem which leads to an inequality?

PERFORMANCE OBJECTIVES: The students will be able to...

1. State the difference between an equation and an inequality.
2. Translate the conditions of the problem into an inequality.
3. Solve an inequality.
4. Check a member of the solution set in the conditions of the problem.

VOCABULARY: No new terms

MOTIVATION:

Carol needs at least $80 to pay for the senior class trip. She already has $25 saved. She can earn $5 an hour working after school. How many hours must she work to be able to pay for the trip?

DEVELOPMENT (Include pivotal questions):

1. Discuss motivation. Elicit that "at least" should be translated as an inequality.

$$5x + 25 \geq 80$$
$$x \geq 11$$

   Answer: Carol must work at least 11 hours to earn the money for the trip.

2. Ask students to translate:
   a) The maximum value of x is 10.
   b) The minimum value of $5x - 1$ is 9.
   c) The value of $3x + 7$ is at least 72.
   d) The value of $x + 7$ exceeds 50.

3. Ask students to suggest solutions in life that would be expressed in inequalities. Discussion may include the following:
   a) old enough to get a learner's permit to drive
   b) too young to vote
   c) test average below passing

4. Pivotal Question: How does the procedure for solving an inequality differ from the procedure for solving an equation? (Multiplication or division by a negative value reverses the sense of the inequality.)

5. Solve and graph the solution on the number line:
   7 more than 8 times a number is more than 13 less than 3 times the number. (Solution: $x > -4$)

   Emphasize the difference between the phrases "more than" and "is more than", as used in this problem.

6. Consider: If 3 times a number is less than 30, what is the largest integral value of the number?

   $$3x < 30$$
   $$x < 10$$

   Answer: 9 (the largest integer which is less than 10).

APPLICATIONS:

1. The perimeter of a square is at least 100. What conclusion can be drawn about the side of the square?
2. The length of a rectangle is 1 more than twice its width. If the perimeter can be no greater than 44 cm, find the maximum value of the length and the width.
3. If twice a number is subtracted from 27, the result is no greater than 3. What is the minimum value of the number?

SUMMARY:

1. What is the main difference between solving an equation and an inequality?

2. What algebraic statement would denote the fact that a number has
   a) a minimum value of 4?
   b) a maximum value of 7?

HOMEWORK:

1. Solve the following problems assuming that the domain of each variable is the set of integers:
   a) 7 more than 3 times a positive integer is less than 20. Find all the possible numbers.
   b) Two sides of a triangle measure $3\frac{1}{4}$ inches each. What are the possible lengths of the third side? (Whole number of inches only.)
   c) The number lies between $-4\frac{1}{2}$ and $+4\frac{1}{2}$.
   d) The area of a rectangle is less than 5. What are its possible dimensions?
   e) The perimeter of a rectangle is 20 cm. What is the greatest area it can have?

2. The perimeter of a square is at most 60 in. What is the greatest length that a side can have?

3. If three times a number is increased by two and the result must not be greater than 10, what is the greatest possible value this number can have?

4. The minimum value of 29 - 2x is 15. What conclusion can you make about the value of x?

5. Solve: $7 - .2x \leq 5$

Lesson 44 - Introduction to Probability

AIM: What do we mean by probability?

PERFORMANCE OBJECTIVES: The students will be able to...

1. Define experiment, event, outcome, sample space, empirical probability, theoretical probability.
2. Conduct a random experiment - toss a coin, choose a card from a standard deck, roll a die.
3. Define probability of certainty and impossible events.
4. Outline uses of probability.

VOCABULARY: experiment, empirical study, sample space, trial, event, outcome, probability, certainty, impossibility, tabulate

MOTIVATION:

Two dice, one red and the other green, are rolled. Jack wins if either 1 or 2 dots turn up on at least one of the dice. Otherwise, Henry wins. Who has the better chance of winning, Jack or Henry?

DEVELOPMENT:

1. Ask: What do you think the answer is?
   Play the game for a while, using red and green dice. (If separate pairs are available, the teacher may wish to divide the class into groups.) Record the outcomes.
   Ask: Does the game work as we expected?
   Pivotal question: How could we use mathematics to analyze this situation?
   Elicit: A die has six sides, each equally likely to come up; this leads to an expectation about the outcome for rolling a die.
   Elicit: A chart would be useful to analyze the possible outcomes of rolling two dice.

2. Display on the chalkboard or distribute rexograph sheets:

	6	.	.A	.	.	.	.C
	5	.	.	.	.	.	.
Green	4	.	.	.	.	.	.
Die	3	.	.	.	.	.D	.
	2	.	.	.	.	.	.B
	1	.	.	.	.	.	.
		1	2	3	4	5	6
				Red Die			

a) How many dots are shown?

b) What does each dot represent?

c) What outcome does each of the following dots represent: A, B, C, D?

d) Mark the dot "E" for the outcome:

$$\begin{cases} \text{Red Die} = 1 \\ \text{Green Die} = 2 \end{cases}$$

e) If an empirical experiment were conducted, for a large number of trials, what part might you expect to turn up with outcome A? ...outcome B? Any of the 36 possible outcomes? ($\frac{1}{36}$)

3. Let: r = number for red die
g = number for green die

a) Draw a ring around the set of dots for which r = 1 or r = 2 or g = 1 or g = 2. How many dots are in the ring? (20)

b) Define probability of an event as the number which expresses the fraction of times we expect that the event will occur.

c) What is the probability that Jack will win? ($\frac{20}{36}$ or $\frac{5}{9}$) Henry? ($\frac{4}{9}$)

d) Does Jack or Henry have the better chance of winning? (Jack, since $\frac{5}{9} > \frac{4}{9}$.)

e) Ask: Did "Jack" win exactly $\frac{5}{9}$ of the time when we played the game today? How can we explain the inconsistency between the empirical (experimental) result and the theoretical prediction?

4. Ring the set of dots for which r = g. What is the probability that r = g?
We write $P(r = g) = \frac{6}{36}$ or $\frac{1}{6}$.

5. Ring the set of dots for which r + g = 5. What is the probability that r + g = 5? $P(r + g = 5) = \frac{4}{36}$ or $\frac{1}{9}$.

6. Find:

	Answers
a) $P(r + q = 6)$	$\frac{5}{36}$
b) $P(r + q = 15)$	$\frac{0}{36} = 0$ (impossible)
c) $P(r + q) \leq 12)$	$\frac{36}{36} = 1$ (certain)

Note: In all cases, $0 \leq$ probability ≤ 1. Probability = 0 denotes impossibility. Probability = 1 denotes certainty.

MATERIALS: red and green dice

APPLICATIONS:

1. $P(r - g = 1) = ?$
2. $P(r + g = 1) = ?$
3. $P((r = 1) \vee (g = 1)) = ?$
4. $P((r = 1) \wedge (g = 1)) = ?$
5. $P(r > 4) = ?$

SUMMARY:

1. What is probability?
2. If a number is selected at random from the set 1,2,3,4,5,6,7 , what is the probability that it will be:
   a) 4
   b) even
   c) less than 3
   d) less than 10
   e) a multiple of 8
3. In today's world, who should be interested in probability?

HOMEWORK:

1. Roll a single die 30 times and keep a record of the outcomes. If you do not have a die, use a numbered hexagonal pencil.
2. Draw the sample space for rolling two dice, a red die and a green die. Ring the dots corresponding to the indicated event and determine its probability.
   a) $P(r + g = 7)$ b) $P(r = 2g)$
   c) $P((r > 4) \wedge (g > 4))$ d) $P(r < 7)$
3. Consider the experiment of tossing a coin. What probability would you assign to having a head turn up? ...a tail turn up? Toss a coin twenty times and keep a record of the outcomes.
4. For the experiment of rolling two dice, find:
   $P(r^2 + g^2 \leq 25)$
5. For the experiment of tossing 4 pennies, what is the probability that 2 will turn up heads and 2 tails? Do this 20 times and keep a record of the outcomes.

Lesson 45 - Biased and Unbiased Objects

AIM: How do we distinguish between biased and unbiased objects?

PERFORMANCE OBJECTIVES: The students will be able to...

1. Determine whether an object is fair or biased upon inspection of the object and empirical evidence.
2. Provide examples of unbiased objects.
3. Conclude that mathematical study of theoretical probability involves only unbiased objects.

VOCABULARY: die (singular of dice), urn, tetrahedron, regular polyhedra, bias, biased, unbiased, theoretical probability, random

MOTIVATION:

A jar contains 4 green marbles, 2 white marbles, 3 blue marbles and one yellow marble. (All marbles are same size.) What is the probability that a blindfolded volunteer will choose a blue marble?

DEVELOPMENT (Include pivotal questions):

1. Suppose that all the blue marbles were smaller than marbles of any other color. Would the probability be affected? How?
2. A game (popular with some students) called "Dungeons and Dragons" uses "dice" which are actually regular polyhedra.
3. Show and discuss several objects which are "unbiased". (pair of dice, standard deck of cards, urn, spinner, digits or letters of the alphabet chosen at random)
4. Elicit responses to simple probability questions, such as the following:
   a) What is the probability of choosing a Jack from an unmarked deck of well shuffled cards?

   b) What is the probability of selecting a daily three-digit lottery number?

5. Show and discuss some examples of biased objects: conical paper cup, weighted coin, two-headed coin, marked deck of cards, loaded dice, thumb tack.

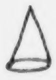
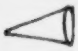
6. Elicit responses to questions such as the following:
   a) What is the probability of choosing an ace if aces are marked in a deck of cards?
   b) What is the probability of getting a tail when tossing a coin which has two heads?

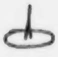
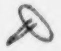
MATERIALS:

Dice (cube or tetrahedron - 4 equilateral triangle faces), standard deck of cards, urn (suggest small red plastic one that comes filled with cheese), spinner from a board game, conical paper cup, weighted coin, marked deck of cards, thumb tack.

APPLICATIONS:

1. Suppose you know the first two digits of a 3-digit number. What is the probability of choosing 2 for the third digit?

2. Toss a conical paper cup 10 times. Call  "up" and  "down". Determine the probability of "up". Is this reliable? Why?

3. Toss a thumbtack. Call  "up" and  "down". Determine the probability of "up". Is this reliable? Why?

SUMMARY:

1. What assumptions did we make yesterday about the dice used by Henry and Jack for their game?
   Elicit: In the early study of probability, all outcomes are assumed to have the same chance of occurring. We say that these outcomes are "equally likely".

In the study of mathematical probability, we concern ourselves only with fair or unbiased objects (and unbiased experimenters).

2. What factors in today's motivation problem ensured that the marble selection was "random"?

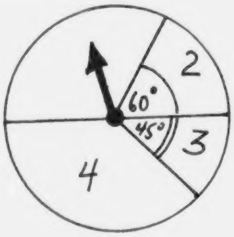
HOMEWORK:

1. The figure below may be used for the experiment of rolling a die and tossing a coin.

Coin						
T	°	°A	°	°	°	°C
H	B°	°	°	°	°	°
	1	2	3	4	5	6

Die

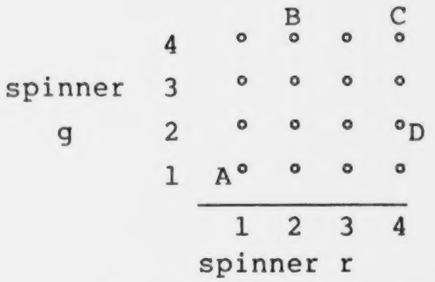
   a) What outcome does each of the following dots stand for: A ___, B ___, C ___?
   b) How many possible outcomes are there for this experiment?
   c) What probability would we normally assign to each outcome?
   d) Find: $P(A \vee B)$? $P(A \wedge B)$?
   e) Find: $P(3 \vee T)$? $P(3 \wedge T)$?



2. For a fair experiment, what probability would you assign to the event that the spinner will stop at 2 which we label P(2)? Now find each of the following:

   a) P(3)
   b) P(4)
   c) P(1)
   d) $P(2 \vee 3)$
   e) $P(1 \vee 3)$
   f) $P(2 \vee 4)$
   g) $P(n>1)$
   h) $P(n<1)$
   i) $P(n<5)$

3. Suppose we had two spinners exactly like the one in example 2, we'll call them red and green. Our experiment will consist of using both spinners. We may picture the sample space for this experiment as in example 1.



a) What outcome does each of the following dots stand for? A ___, B ___, C ___, D ___

b) How many possible outcomes are there for this experiment?

c) What probability would you assign to each outcome? (Note that the answer should not be $\frac{1}{16}$.)

d) Find:

1) $P(r + g = 8)$ 2) $P(r + g = 9)$

3) $P(r + g = 2)$ 4) $P(r + g = 3)$

5) $P\left[(r=1)\vee(g=1)\right]$ 6) $P(r + g = \text{a prime})$

e) What assumptions are we making regarding the experiment in order to obtain probabilities?

4. Toss 2 coins 100 times and record the following information:

a) No. of outcomes showing 2 heads ____

b) No. of outcomes showing 2 tails ____

c) No. of outcomes showing a head and a tail ____

What probability would you normally assign to each of the 3 events? Are these the same as your empirical results? Explain.

5. A coin is tossed 10 times and comes up heads each time. What is the probability that the next outcome will also be a head?

6. Challenge problem (the famous "birthday problem"): How many people must be in a room so that the probability that two (or more) share the same month and date of birth is greater than 50%?

Some tips:

a) $N > 23 \longrightarrow 50\%^{+}$

b) $N > 34 \longrightarrow 80\%^+$

c) $N > 46 \longrightarrow 95\%^+$

d) Be sure all in room are counted and keep a tally.

Lesson 46 - Simple Probabilities

AIM: How do we calculate simple probabilities?

PERFORMANCE OBJECTIVES: The students will be able to...

1. Define uniform probability and random selection.
2. Write and apply the fundamental probability formula: $P(E) = \frac{n(E)}{n(S)}$.
3. Provide examples requiring use of the probability formula.

VOCABULARY: theoretical possibility, sample space, outcome, event, random selection, uniform probability, continuum

MOTIVATION:

A standard deck of 52 cards is shuffled. A single card is drawn at random. What is the probability that the card drawn is: 1) a seven 2) a red three 3) any black card?

DEVELOPMENT (Include pivotal questions):

Develop the following concepts with the class.

1. Theoretical probability is used for objects known to be unbiased. Otherwise we must use empirical study.
2. Outcome is the result of some activity.
   Example: When rolling a die, the outcome is a number from 1 to 6 inclusive.
3. Sample space is the set of all possible outcomes of an activity.
   Example: Choose one card from a deck of 52 unmarked, well shuffled cards. The sample space in all 52 cards identified by denomination and suit.
   Note: n(S) is the number of elements in the sample space. In this case, n(S) = 52.

4. Event is a subset of the sample space (must be clearly defined).

   Example: A roll on a single toss of die produces a number greater than 3. "Event" must be distinguished from "outcome". Note: n(E) is the number of elements in the event. Note that an outcome of 4, 5 or 6 is favorable for this event. In this case, n(E) = 3.

5. Theoretical probability defined: Probability of an event (E) is the number of ways for a favorable outcome to occur divided by the number of outcomes in the sample space. $P(E) = \frac{n(E)}{n(S)}$. (Assuming all outcomes are "equally likely".)

   a) Probability of an impossible event is 0.

      Example: Choose the letter x from among M,A,T,H.

   b) Probability of a certainty is 1.

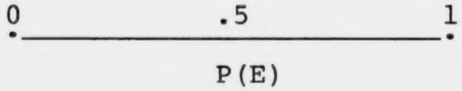
      Example: Choose a consonant from T,R,H,S.

      Example: Choose a vowel from A,E,I,O,U.

   c) For any event E, P(E) is between 0 and 1 inclusive.

      Symbolically: $0 \leq P(E) \leq 1$

      In other words, P(E) falls somewhere on this continuum.

   d) Probability can be expressed as a ratio, fraction, decimal or percent.

   e) The sum of the probabilities of any set of events that partition the sample space for any experiment is 1.

      Example: The dice game played by Jack and Henry in Lesson 44.

6. Uniform probability is the expectation that all outcomes are equally likely.

   Examples:

   Flipping a coin: H or T.

   Birth of a child: boy or girl.

   Tossing a die: any outcome of 1 to 6 is equally likely.

7. Random selection means that there is no tampering or interference on the part of the experimenter or anyone else.

MATERIALS: deck of cards, dice, coins

APPLICATIONS:

1. Consider a spinner (from a board game) with 10 congruent regions. Unbiased, theoretical probability applies. The arrow must point only to one region (if on line, do over).
    a) What is n(S)?
    b) Calculate P(7)
    c) Event: Any even number greater than 6. Calculate P(E).
    d) Event: Any number less than 5. Calculate P(E).
    e) Event: Any odd number. Calculate P(E).

2. A card is chosen from a well shuffled deck. Find:
    P(any ace)
    P(red Jack)
    P(black card)
    P(any card except 7)

3. A fair die is rolled. Find:
    P(even #)
    P(prime #)
    P(#>1)
    P(5 or even #)

4. Some Easter eggs are in a basket, colored as follows: 4 green, 2 blue, 1 yellow. An egg is randomly selected. Find:

P(G)	P(not blue)
P(not yellow)	P(white)
P(yellow)	P(not orange)

SUMMARY:

1. What is theoretical probability?

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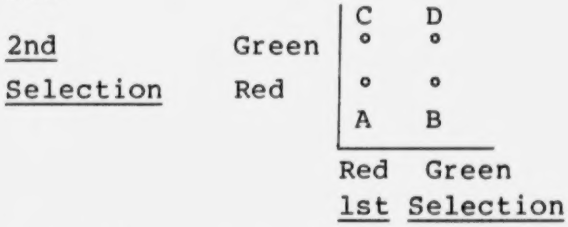
2. State the fundamental probability formula.

3. What are the restrictions on the number which expresses a probability?

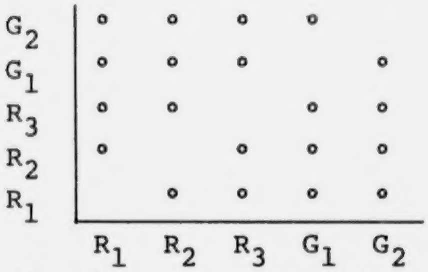
HOMEWORK:

1. A bag contains 3 red marbles and 2 green marbles, indistinguishable except for color.
   a) If a single marble is selected at random, what is the probability that it will be:
   1) red 2) green?
   b) If a marble is selected, returned and again a marble is selected, we may show the sample space as:



   (1) What is the probability of each pair: A,B,C,D?
   (2) Viewed this way, are the 4 outcomes equally likely?
   (3) How can we show the sample space to reflect uniform probability?

   c) Suppose now that 2 marbles are selected at the same time. We may picture its sample space also.



   (1) Why are subscripts shown?
   (2) Why are the diagonal dots omitted?
   (3) How many possible outcomes are there?
   (4) What is the probability that
      a) both marbles are red?
      b) both marbles are green?

c) one marble is red and the other is green?

(5) If we make a 2 marble selection one at a time, without returning, what is the probability that
   a) The 1st marble is red and the 2nd green?
   b) The 1st marble is green and the 2nd red?
   c) The 1st marble is red or the 2nd is green?

2. Refer to exercise 3 of previous homework.
   a) Find these probabilities:
      1) $P(r = g)$
      2) $P(r > g)$
      3) $P(r < g)$
      4) $P(r = 2g)$
      5) $P(r > 1 \wedge g > 1)$
      6) $P(r > 1 \vee g > 1)$
   b) Picture its sample space in a manner that reflects uniform probability.

3. A card is chosen at random from a bridge deck of 52 cards. What is the probability that this card is:
   a) a picture card.
   b) an ace and red.
   c) an ace or red.
   d) a picture card or red.
   e) not a picture card.
   f) neither a picture card nor red.

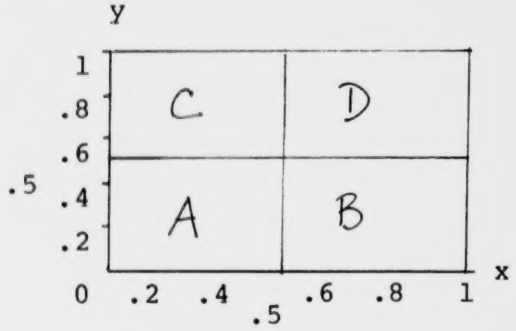
4. A penny is tossed and a die is rolled. Suppose that the head outcome stands for 2 while the tail outcome stands for 3.
   a) Picture the sample space for this experiment.
   b) What is the probability that the sum will be:
      1) 1
      2) 2
      3) 3
      4) 4
      5) 5
      6) 6
      7) 7
      8) 8
      9) 9
      10) 10
      11) more than 5
      12) more than 2

5. An English test consists of 25 multiple-choice questions with 5 alternates for each question. For each incorrect response there is a penalty of 1 point. If the question is not answered there is no credit and no penalty.
   a) Is the penalty justified? Why?
   b) Under what circumstance is it wise to guess?

c) Suppose the test consisted of only 20 multiple choice questions. Is the 1 point penalty justified? Why?

6. A computer is used to generate random numbers in the interval from 0 to 1. Two such numbers are used to obtain the x and y coordinates for locating a point. What is the probability that a pair of random numbers will locate a point in region

a) A
b) B
c) C
d) D



7. How can the computer be used to find the area of region A in exercise 4? If you have access to a computer, check to see how close your answer agrees with P(A).

*8. Challenge Problem: Four men go into a night club and check their hats. the hat check person is fired from her job during the evening and absconds with the checks. What is the probability that exactly three men get their correct hats back.

9. Challenge problem: TRISKAIDEKAPHOBIA - fear of the # 13. FRIDAY THE THIRTEENTH day of the month occurs 48 times every 28 years. What is the probability that any given Friday falls on the 13th?

Two acceptable answers:

using 365 day year $\frac{48}{1460}$
using 365.25 day year $\frac{48}{1461}$

Lesson 47 - Probability of "A and B"

AIM: How do we calculate the probability of "A and B"? P(A B)

PERFORMANCE OBJECTIVES: The students will be able to...

1. Review the logical connective "and". Review the set operator $\cap$ (intersection).
2. Apply the principal that "p and q" is true if p is true and q is true.
3. Given events A and B, calculate P(A), P(B) and P(A and B), i.e., $P(A \wedge B)$.

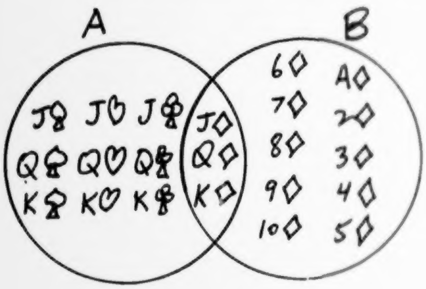
VOCABULARY: set, intersection, union, disjoint, empty (or null) set

MOTIVATION:

One card is randomly drawn from a standard deck. What is the probability that it will be a face card and a diamond?

DEVELOPMENT (Include pivotal questions):

1. Review, as necessary, operations with sets including union, intersection, disjoint set, etc.
2. Recall from the logic unit that the connective "p and q" is written symbolically as "$p \wedge q$" and that "$p \wedge q$" is true only when p is true and q is true.
3. Consider motivation exercise:



A = Set of face cards K,Q,J

B = Set of diamonds A◊, 2◊, 3◊... K◊

In set notation, $A \cap B$ means all elements simultaneously in A and in B.

P(face card and ◊) = $\frac{3}{52}$

4. On one roll of a die, what is the probability of rolling a number that is odd and less than 5?

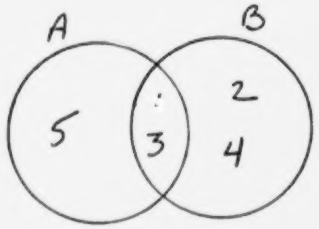
   set A: odd #, set B: #<5

   P(odd and less than 5) = $P(A \cap B) = \frac{2}{6}$

   Ask: Why not $\frac{2}{5}$?

   Answer: Because n(S) = 6

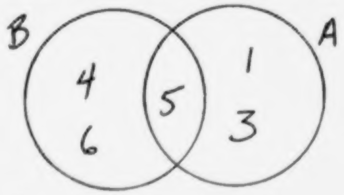
   (i.e., 1,2,3,4,5,6)



5. On one roll of a die, what is P (the number is odd and greater than 3)?

   set A: odd, set B: >3

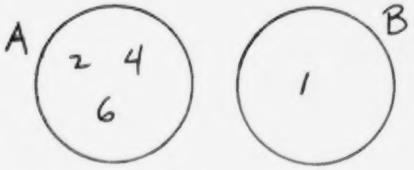
   $P(A \cap B) = \frac{1}{6}$



6. On one roll of a die, what is P (the number is even and less than 2)?

   set A: even, set B: <2

   $P(A \cap B) = 0$



   This is impossible. A and B are disjoint sets.

Conclusion: No simple formula will yield $P(A \cap B)$. Each case must be examined individually.

MATERIALS: standard deck of 52 cards, pair of dice, overhead projector for set operations as needed

APPLICATIONS:

1. On one roll of a die, what is:
   a) P(even and <6)
   b) P(odd and 4)
   c) P(even and 6)
2. If a card is drawn at random from a deck, find:
   a) P(face card and red)
   b) P(red and ace)
   c) P(red and black)
3. If an integer is drawn at random from the set of integers from 1 to 20 (inclusive), find:
   a) P(odd and greater than 8)

b) P(odd and prime)
c) P(odd and a multiple of 3)
d) P(multiple of 3 and multiple of 5)
e) P(multiple of 3 and factor of 100)

SUMMARY: Elicit $P(A\ B) = \frac{n(A \cap B)}{n(S)}$

Note: No simple formula works for all cases. We must observe the intersection of the two sets and count the number of elements in the intersection.

HOMEWORK:

1. A positive integer less than 21 is chosen at random. Let the following letters denote number properties:

o = odd
e = even
p = prime
c = composite
u = perfect square
h = greater than 10

Find the following probabilities:

a) $P(o)$
b) $p(e)$
c) $P(p)$
d) $P(c)$
e) $P(u)$
f) $P(h)$
g) $p(o \wedge p)$
h) $p(o \wedge e)$
e) $P(o \wedge h)$
f) $P(e \wedge h)$
g) $P(c \wedge h)$
h) $P(p \wedge h)$
i) $P(e \wedge p)$
j) $P(p \wedge c)$
k) $P(u \wedge h)$
l) $P({\sim} u \wedge h)$
m) $P(o \wedge c \wedge h)$
o) $P({\sim} u \wedge {\sim} h)$

2. Using the abbreviations in example 1, and the set of positive integers less than 21 for domain, find:

a) $P((2x + 3 > 9) \wedge o)$
b) $P((2x + 3 > 9) \wedge e)$
c) $P((2x + 3 > 9) \wedge {\sim} h)$
d) $P([y + 5 < 2(3y - 10)] \wedge p)$

3. Let two statements, p and q, be chosen at random. Find these probabilities.

a) $P(p \wedge q$ is true)
b) $P(p \vee q$ is true)
c) $P(p \longrightarrow q$ is true)
d) $P(p \longleftrightarrow q$ is true)

*4. Given $P(A) = .4$, $P(B) = .3$, $P(A \cap B) = .2$, find $P(A \cup B)$.
(Answer: $P(A \cup B) = .5$)

Lesson 48 - Probability of "A or B"

AIM: How do we calculate the probability of "A or B"? P(AUB)

PERFORMANCE OBJECTIVES: The students will be able to...

1. Review the logical connective "or".
   Review the set operator U (union).
2. Given P(A) and P(B), derive and apply the formula
   $P(A \cup B) = P(A) + P(B) - P(A \cap B)$
   to intersecting and disjoint sets.

VOCABULARY: union of sets, disjoint sets

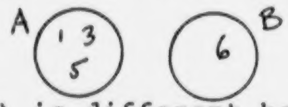
MOTIVATION:

On a single toss of a die, calculate the probability of rolling:

a) an odd number and a number greater than 5.
b) an odd number or a number greater than 5.

DEVELOPMENT (Include pivotal questions):

1. Consider motivation problems.
   Ask: What is the roster of the event: odd number?
   set A = $\{1,2,5\}$
   What is the roster of the event: greater than 5?
   set B = $\{6\}$
   Do problems a) and b) have the same answer?
   Elicit: Problem a), similar to those considered yesterday, involves the intersection of these sets (which is the null set, since they are disjoint).



Problem b) is different because it involves the connective "or". Students should observe that the event for problem (b) is $\{1,3,5,6\}$ which is the union of set A and set B.

Since $n(S) = 6$, the answers to the motivation problems are:

a) $P = \frac{n(A\ B)}{n(S)} = \frac{0}{6} = 0$ (impossibility)

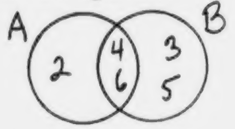
b) $P = \frac{n(A \cup B)}{n(S)} = \frac{4}{6}$ or $\frac{2}{3}$

2. Use the procedure outlined in 1 to calculate the probability of rolling an even number or a number greater than two on a single toss of a die.

Roster of A, B: $A = \{2,4,6\}$, $B = \{3,4,5,6\}$

Roster of AUB: $A \cup B = \{2,3,4,5,6\}$

Venn diagram:

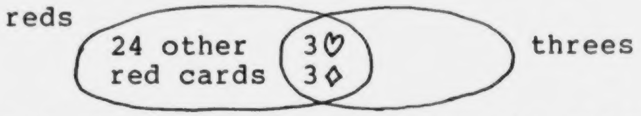


$P(E) = \frac{5}{6}$

3. In logic, "pVq is true" means "p is true or q is true". In sets, an element is in PUQ if it is in P or in Q or in both sets.

4. Consider P (red card or a 3) on drawing one card from a standard deck.

a) Examine the Venn diagram and conclude that $P = \frac{28}{52}$.



b) Remind the students that yesterday the class concluded that a formula for $P(A \cap B)$ could not be found. Suggest that there might be a formula for $P(A \cup B)$.

c) Elicit: A strategy which might lead to the discovery of the formula is to list known facts about this situation and see if a relationship can be observed.

let R : red card, T : three

Elicit the values:

$n(R) = 26 \qquad P(R) = \frac{26}{52}$

$n(T) = 4 \qquad P(T) = \frac{4}{52}$

$n(R\ T) = 2 \qquad P(R\ T) = \frac{2}{52}$

$n(R \cup T) = 28 \qquad P(R \cup T) = \frac{28}{52}$

The teacher may wish to draw a dotted line as indicated to emphasize the goal.

Students may observe (incorrectly) that $P(R) + P(R \cap T) = P(R \cup T)$. This may be disproved by attempting to apply it to another situation (such as the motivation problem).

The correct relationship is

$$\frac{26}{52} + \frac{4}{52} - \frac{2}{52} = \frac{28}{52}$$

5. Students should memorize the formula:

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

MATERIALS: pair of dice, deck of 52 playing cards

APPLICATIONS:

1. Find the probability of drawing a 7 or a diamond or a single draw from a deck of cards $(\frac{16}{52})$.
2. Find the probability of drawing a diamond or a three on a single draw from a deck of cards.
3. Roll a single die. Find the probability of tossing an even number or a number less than 2. Since $A \cap B = \emptyset$, $P(A \cap B) = 0$ $\quad P(A \cup B) = P(A) + P(B) = \frac{3}{6} + \frac{1}{6} = \frac{4}{6}$
   Generalize this result (for disjoint sets).
4. Challenge problem: A letter of the alphabet is chosen at random. Find the probability that the letter selected:
   a) is a vowel
   b) is a consonant
   c) precedes u (in alphabetical order) or is a vowel
   d) follows t and is a vowel
   e) follows u and is a vowel

SUMMARY:

1. What is the probability of "A or B"?
   Elicit statement of formula:

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

2. What comparison symbol would you insert between each pair?
   a) $P(A)$ ____ $P(A \cup B)$ b) $P(A)$ ____ $P(A \cap B)$

HOMEWORK:

1. Let S be the set of positive integers less than 11.

$A = \{1,3,5\}$

$B = \{2,4,6,8\}$

$C = \{4,5,6,7\}$

A number is selected at random from S. Assuming uniform probability (Note: $P(A)$ is the probability the number is in A) find:

a) $P(A)$
b) $P(B)$
c) $P(C)$
d) $P(B \cap C)$
e) $P(B \cup C)$
f) $P(A \cap C)$
g) $P(A \cup C)$
h) $P(A \cup B)$
i) $P(A \cup B)$
j) $P(A \cap B)$

2. For the same experiment as in example 1, find:

a) $P(A \cup (B \cap C))$
b) $P((A \cup B) \cap (A \cup C))$
c) $P(B \cup (A \cap C))$
d) $P((B \cup A) \cap (B \cup C))$
e) $P(\sim A)$ (This is the probability that the number is not in A.)
f) $P(\sim A \cap \sim B)$
g) $P(\sim (A \cup B))$

3. Let a positive integer, n, less than 21 be randomly selected. Find:

a) $P(2n > 10 \lor n < 10)$
b) $P(2n < 5 \land n > 10)$
c) $P(2n < 5 \lor n > 10)$
d) $P(3(5 - n) < n - 1)$

4. For the same experiment as in example 1, find:

a) $P(A \cup B \cup C)$

*b) a formula corresponding to

$P(A \cup B) = P(A) + P(B) - P(A \cap B)$

for $P(A \cup B \cup C)$. Check your answer by

1) using previous information
2) a Venn diagram

Lesson 49 - Probability of a Complement, $\bar{A}$

AIM: What do we mean by the probability of "not A"?

PERFORMANCE OBJECTIVES: The students will be able to...

1. Define $\bar{A}$ as the complement of A.
2. State and apply principles:
   a) The probability of an event is equal to the sum of the probabilities of all the singleton outcomes in the event.
   b) The sum of the probabilities of all possible singleton outcomes for any sample space must always equal 1.

VOCABULARY: complement of a set, complement of an event, universal set

MOTIVATION:

In selecting a card at random from a standard deck, what is the probability of selecting

a) a face card (K,Q,J)?
b) a card that is not a face card?

DEVELOPMENT (Include pivotal questions):

1. Discuss motivation problem.

   Answers: a) $\frac{12}{52}$ b) $\frac{40}{52}$

   Ask: What is the relationship between conditions a) and b)?

   Students should observe that these are mutually exclusive possibilities that together account for all possible outcomes. In other words, if set F: $\{$face card$\}$ and set G: $\{$not a face card$\}$, then F and G are disjoint events and $F \cup G = S$ (the sample space). Set G is called the complement of set F, and is designated by the symbol $\bar{F}$, read "the complement of F".

Students should also observe that $P(F) + P(\overline{F}) = 1$, and that this must always be true for an event and its complement.

Elicit that this principle may be used to calculate $P(\overline{F})$ readily when $P(F)$ is known. Some students may have used subtraction ($1 - \frac{12}{52} = \frac{40}{52}$) in solving motivation problem b).

2. Given the set of single digit numbers $D = \{0,1,2,3,4,5,6,7,8,9\}$ as the universal set, write the roster of the complement of:
   a) $S = \{6,7\}$ $\quad\overline{S} =$
   b) $E = \{0,2,4,6,8\}$ $\quad\overline{E} =$
   c) $D = \{\ \}$ $\quad\overline{D} =$

3. An urn contains 4 black, 3 white, 2 blue and 1 red marble. One marble is selected at random. Calculate:
   a) P(BLK) d) P(WH) f) P(YEL)
   b) $P(\overline{BLK})$ e) $P(\overline{WH})$ g) $P(\overline{R})$
   c) $P(\overline{ORANGE})$

4. From the word "probability" select any letter at random. Calculate:
   a) $P(\overline{F})$
   b) P(not a vowel)
   c) P(not i)

APPLICATIONS:

1. Start with only 4 cards from a standard deck:

   K , Q , J , 10 .

   Let $A = \{K\}$. Then $\overline{A} = \{Q , J , 10\}$.

   Have students make the following calculations:

   $P(A) = \frac{1}{4}$ $\quad P(\overline{A}) = \frac{3}{4}$ $\quad P(A) + P(\overline{A}) = 1$

2. A spinner has 8 congruent regions numbered 1 through 8. Calculate:

   P(prime) P(not prime)
   P(4) $P(\overline{4})$
   P(odd) $P(\overline{odd})$

3. What is the probability that the current U.S. president is not an American citizen?

4. From a standard 52 card deck, find $P(\overline{K \text{ or } Q})$ on a draw of one card.

$P(K \text{ or } Q) = \frac{4}{52} + \frac{4}{52} = \frac{8}{52}$

$1 - \frac{8}{52} = \frac{44}{52}$

The probability of drawing a red card is $\frac{44}{52}$.

SUMMARY:

1. $P(A) + P(\bar{A}) = 1$. What does this formula mean?
2. The probability of rain tomorrow is $\frac{1}{3}$; the probability of snow is $\frac{1}{9}$; what is the probability that there will be neither rain nor snow?

HOMEWORK:

1. Let: $S = \{1,2,3,4,5,6,7,8,9,10\}$

$A = \{1,2,5,6\}$

$B = \{2,4,6,7\}$

$C = \{4,5,6,7\}$.

A number is selected at random from S. Find:

a) $P(A)$ and $P(\bar{A})$ (Note that $P(A)$ is the probability that the number was selected from A.)

b) $P(B)$ and $P(\bar{B})$

c) $P(AUB)$

d) $P(\bar{A}U\bar{B})$

e) $P(B\ C)$

f) $P(\bar{B}U\bar{C})$

2. A card is selected at random from the 13 diamonds of a 52 card bridge deck. Find:

a) P(it is a picture card)

b) P(it is not a picture card)

c) P(it is not an even number)

d) P(it is an odd number) (assume ace = 1)

e) P(it is not an odd number)

3. A card is selected at random from a full 52 card bridge deck. Find:

a) P(a King or an ace)

b) P(a King or a diamond)

c) P(neither a King nor a diamond)

d) P(a Queen or a King)

e) P(neither a Queen nor a King)

4. Express $P(\overline{A \cup B})$ in terms of $P(A)$, $P(B)$, $P(A\ B)$.

Lesson 50 - The Counting Principle

AIM: How do we use the counting principle to determine n(S), the size of sample spaces?

PERFORMANCE OBJECTIVES: The students will be able to...

1. Write the definition of the counting principle.
2. Apply the counting principle to problems involving menu selections, rolls of dice, urn problems. (Calculator applications.)
3. Use tree diagrams to illustrate the counting principle.

VOCABULARY: counting principle, tree diagram

MOTIVATION:

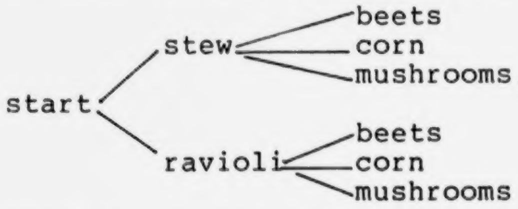
Today the cafeteria offers:

MAIN COURSE	VEGATABLE
stew	beets
or	corn
ravioli	mushrooms

If you must choose one vegetable with your main course, list all of the possible lunch combinations.

DEVELOPMENT:

1. Discuss motivation problem. Show how a tree may be used to ensure that all six possibilities are considered.



Elicit that the number of possible combinations may be calculated by multiplying the number of main course offerings by the number of vegetable offerings.

State the counting principle: If one activity can occur in m ways and following this a second activity can occur in n ways, then the two activities can occur in the order given in mn ways.

2. Generalize the above example by asking: Suppose you are also given a choice of five desserts; how many complete lunch-combinations are now possible?
   Elicit that the tree could be extended with five new branches from each old branch tip; thus, the answer is $2 \cdot 3 \cdot 5 = 30$.
   Generalize the counting to apply to longer sequences of activities.

3. Consider: How many different license plates can be formed from the 26 letters of the alphabet and the ten digits using:
   a) 1 letter followed by 4 digits?
   b) 3 numbers followed by 3 letters?
   c) 4 numbers followed by 3 letters?
   (This motivation exercise actually reflects the recent history of license plates in New York State.)
   a) $26 \cdot 10 \cdot 10 \cdot 10 \cdot 10 = 260{,}000$ (Used widely in the 50's and 60's, with some arrangements omitted: X0007, etc.)
   b) $10 \cdot 10 \cdot 10 \cdot 26 \cdot 26 \cdot 26 = 17{,}576{,}000$ (Used through most of the 70's, with no meaningful three letter words such as "NUT", "COW", etc. permitted.)
   c) $10 \cdot 10 \cdot 10 \cdot 10 \cdot 26 \cdot 26 \cdot 26 = 175{,}760{,}000$ (In current use.)
   Note that there were many forbidden plates.
   Example: DPL, FC, SEX, etc.
   Now we can also have vanity plates using up to eight letters.
   On a Mercedes: R U NVUS

APPLICATIONS:

1. A menu offers 5 main courses, 3 beverages and 7 desserts. How many different meals can be ordered including:

   a) 1 main course, 1 beverage, 1 dessert?
   b) 1 main course, 1 beverage, no dessert?
   c) dessert and beverage only?

2. A girl has 4 sweaters and 5 skirts. How many different outfits (sweater and skirt) are possible? Assume no color clashes.

3. A student has a true-false test with ten questions. How many different answer keys are possible? (Hint: If you have trouble with this, try it first with 3 questions as an introduction.)

4. Baskin-Robbins offers 31 flavors of ice cream. How many different double scoops can be made if the two flavors may not be the same?

SUMMARY:

1. State the counting principle.
2. How many 7-digit phone numbers are possible (assuming that the number cannot start with zero)? Does this explain why some metropolitan areas require several area codes?

HOMEWORK:

1. How many two-letter "words" are there that begin with a vowel and end with a consonant?
2. How many two-letter words are there with no repeated letter?
3. How many two-digit numbers have a prime for the tens place?
4. How many two-digit numbers have an odd number for the tens digit and an even number for the units digit?
5. How many three-digit numbers have an odd digit in the hundreds place?
6. How many three-digit numbers have a prime in the hundreds place, an odd number in the tens place and an even number in the units place?

7. A man has 3 ways of traveling from A to B and 5 ways of going from B to C. How many ways does he have to make the trip from A to C by way of B? List all these ways.

8. A class consists of 12 boys and 13 girls. How many President, Vice President election pairs are possible if the pair must be of opposite sex? If the vice president must be a girl?

Lesson 51 - Compound Events

AIM: How do we calculate the probabilities of compound events?

PERFORMANCE OBJECTIVES: The students will be able to...

1. Write the definition of compound events.
2. Employ the counting principle and tree diagrams to calculate probabilities involving compound events.
3. Solve verbal problems involving compound events.

VOCABULARY: independent events, dependent events, ordered pairs of elements in sample space, ordered triples of elements in sample space, ordered n - triples of elements in sample space

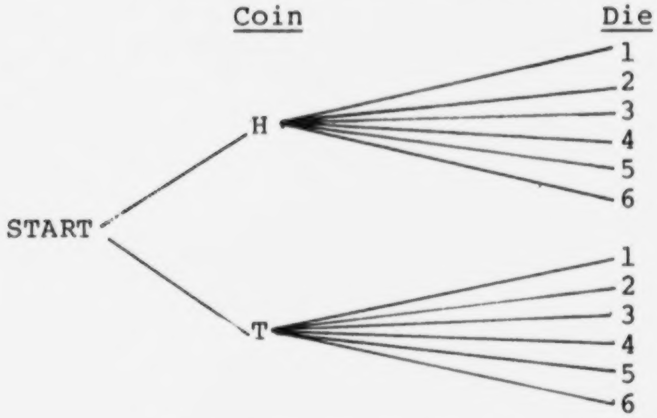
MOTIVATION:

A fair coin is tossed, then a fair die is rolled. What is the probability of a tail on the coin and a 3 on the die?

DEVELOPMENT:

1. Discuss the motivation exercise:
    a) What is the probability of a tail on the coin?
    ($P(T) = \frac{1}{2}$)
    What is the probability of a 3 on the die?
    ($P(3) = \frac{1}{6}$)
    Does the outcome of the first event have any bearing on the outcome of the second event? (No)
    When the outcomes of activities do not effect each other, we call them independent events. An event which consists of two or more independent events is called a compound event.

b) The problem can be solved using a tree diagram:



How many outcomes are possible? (12)

(This may also be obtained by using the counting principle.)

List roster of sample space as ordered pairs:

{(H,1),(H,2),(H,3),(H,4),(H,5),(H,6),(T,1),(T,2),(T,3),(T,4),(T,5),(T,6)}

c) How many outcomes satisfy the conditions of the problem? (1)

$P(E) = \frac{n(E)}{n(S)} \longrightarrow P(T,3) = \frac{1}{12}.$

d) Could we have used another method to reach this result?

Do you observe any relationship between P(T,3), P(T) and P(3)?

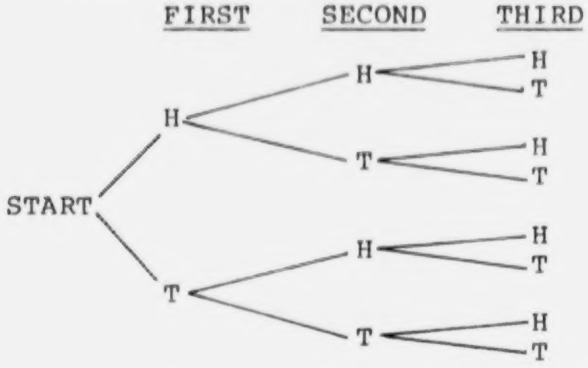
Elicit: $\frac{1}{12} = \frac{1}{2} \cdot \frac{1}{6}$

The probability of the compound event is the product of the probabilities of the separate independent events.

e) Which method is easier to use?

2. Assuming that the probability that a person will finish college is $\frac{1}{4}$ and the probability that a person will become a doctor is $\frac{1}{20}$, is it true that the probability that a person will finish college and become a doctor is $\frac{1}{80}$? Explain.
This is not a compound event because the separate activities are not independent, so the formula does not apply.
Elicit other examples of independent and dependent events.

3. Use a tree diagram to show the sample space in tossing 3 coins simultaneously.



a) What is n(S)? (8)

b) List sample space roster (ordered triplets).

{(H,H,H),(H,H,T),(H,T,H),(H,T,T),(T,H,H),(T,H,T),(T,T,H),(T,T,T)}

c) How was roster constructed?

d) What is the probability of getting 2 tails and a head? $(\frac{n(E)}{n(S)} = \frac{3}{8})$

e) How could you have used the multiplication rule for compound events to answer question d?

On a single coin, $P(T) = \frac{1}{2}$ and $P(H) = \frac{1}{2}$.

Therefore, the probability of getting a head on the first coin and tails on the other two coins is

$\frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{8}$.

Since the head may occur on any of the three coins, this value must be multiplied by 3 to produce the answer. $3 \cdot \frac{1}{8} = \frac{3}{8}$

APPLICATIONS:

1. If a coin is tossed and a die is rolled, what is P(H 5)?

2. If 3 coins are tossed simultaneously, are the tosses dependent or independent? (independent)

Calculate:	Answers:
P(all heads)	$(\frac{1}{8})$
P(at least 2T)	$(\frac{4}{8})$
P(exactly 2H)	$(\frac{3}{8})$
P(at least 1T)	$(\frac{7}{8})$
P(not all heads)	$(1 - \frac{1}{8} = \frac{7}{8})$

3. Calculate the probability of at least 1 boy in a family with 3 children. (Answer $\frac{7}{8}$)

4. If all questions are answered by guessing, compare the probability of answering 3 questions on a true-false test correctly with the probability of answering 3 questions on a matching test (each test has 3 questions).
   Answer: On T-F, P(all correct) = $\frac{1}{8}$
   On matching, P(all correct) = $\frac{1}{6}$
   Which is easier, T-F or matching? (See homework problem #5.)

SUMMARY:

1. How do we evaluate probabilities of compound events?
   Method 1:
   - a) List the sample space by a tree diagram or ordered elements. Count n(S).
   - b) Count n(E), the number of elements that satisfy the given conditions.
   - c) Use formula: $P(E) = \frac{n(E)}{n(S)}$

   Method 2: Multiply the probabilities of the separate independent events.

2. The probability that it will rain tomorrow is $\frac{2}{5}$. the probability that you will have spinach for dinner tonight is $\frac{1}{4}$. What is the probability that you will have spinach tonight and it will not rain tommorow?

HOMEWORK:

1. A coin is tossed 4 times. What is the probability that:
   - a) 4 heads will turn up?
   - b) 3 heads and 1 tail will turn up?
   - c) 2 or more heads will turn up?
   - d) 2 heads and 2 tails will turn up?

   (A tree diagram might help.)

2. 6 coins are tossed. What is the probability that there will be 3 heads and 3 tails?

3. A red die and a green die are rolled. What is the probability that the red die will show an odd number of dots while the green die will have fewer than 5 dots? Show a tree diagram for this experiment. Using your tree, do you get the same answer as using the counting principle?

4. There are 5 spots on the gym floor, 1,2,3,4,5. How many different ways could Amy, Bess and Caroline occupy these spots?

5. Calculate the probabilities of guessing (randomly) all answers correct in a

   a) 6 question true-false quiz. (answer: $\frac{1}{64}$)

   b) 6 question matching test. (answer: $\frac{1}{720}$)

   (Assume that you know none of the answers at the start of the test.)

6. The probability thatGeorge will fail French is .2 and the probability that he will fail math is .25; find the probability that he will pass both courses.

Lesson 52 - Permutations

AIM: What do we mean by permutations?

PERFORMANCE OBJECTIVES: The students will be able to...

1. Define factorial (!) to calculate number of arrangements of n things taken n at a time.
2. Define permutation.
3. Solve problems involving permutations on n things n at a time.

VOCABULARY: permutation, n factorial (notation n!), subscript (as in ${}_nP_n$)

MOTIVATION:

In how many ways can we arrange the order of six songs on one side of a record?

DEVELOPMENT (Include pivotal questions):

1. Discuss motivation problem. An arrangement in which order counts is called a permutation. Elicit: Any one of the six songs could be the first song and any of the five remaining ones be the second. Counting in this fashion, we arrive at $6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 720$ possible permutations.
   This type of calculation is used so frequently that it is given a name. In this case it is 6! Read 6! as "six factorial".
   In general, this is known as n! and is read "n factorial". Many calculators carry this function on a single button.
2. The number of permutations of n different things taken n at a time is n!. Determine how many ways 14 people could seat themselves at a rectangular dinner table.
   Answer: 14! = 87,178,291,200

Note: At the rate of one change per second, all possibilities would not be exhausted for over 2700 years.

$14! \div (60 \times 60 \times 24 \times 365.25)$

3. For practice, evaluate:
   a) $3!$
   b) $\frac{6!}{3!}$ Note: The answer does not equal $2!$
   c) $(5-2)!$
   d) $7! - 4!$

APPLICATIONS:

1. In how many ways can 8 horses cross the finish line in a horse race (exclude ties)?

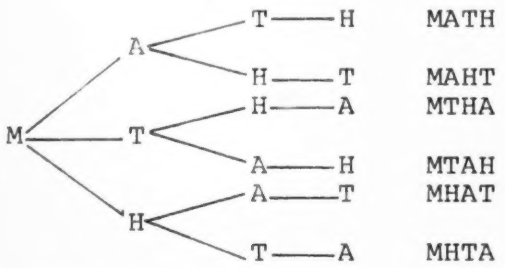
2. In how many ways can the digits 0 through 9 be selected to form a 10-digit number if each digit is used just once? (Answer = $9 \cdot 10^9$ = 9,000,000,000. Note: zero cannot be used as the first digit.)

3. In how many different ways can the letters of MATH be arranged?

   Show tree diagram and list the 24 possibilities.

   MATH
   MAHT
   MHAT
   MHTA
   MTHA
   MTAH
   AMTH
   AMHT
   ATMH
   ATHM
   AHTM
   AHMT
   TMHA
   TMAH
   THAM
   THMA
   TAMH
   TAHM
   HMAT
   HMTA
   HTMA
   HTAM
   HAMT
   HATM

   Tree diagram for permutations on MATH.



etc.*
(Starting
with A,T,H)

*(Fully shown in next lesson.)

4. Use factorial notation to write the number of
   a) seating arrangements for 32 students.
   b) orders of finish for 17,000 marathon runners, assuming no ties.
   c) permutations on the 26 letters of the alphabet.

SUMMARY:

1. What is a permutation?
2. In how many different ways could p people be seated in a row?

HOMEWORK:

1. Compute:

   a) $\frac{(5 + 2)!}{5!}$ b) $\frac{(3 \times 2)!}{(3!)(2!)}$ c) $5! - 4!$

2. How many 6-digit numbers are there if no two digits are the same and the only digits that may be used are: 1,2,3,4,5,6?
3. How many 6-digit numbers are there if 1,3,5 can occupy only odd positions while 2,4,6 can only occupy even positions and no digit is repeated?

$$\underbrace{\overline{1}\ \overline{2}\ \overline{3}\ \overline{4}\ \overline{5}\ \overline{6}}_{\text{positions}}$$

4. How many 5 character codes are possible if the middle character is a letter and the other four characters are the numerals 1,2,3,4 without repetition?
5. Find positive integers $a>b>c$ such that $a! \div b! = c!$
6. How many consecutive zeros, starting from the unit's digit, will the computed value for 18! have? Try to do this without actually computing 18!
7. Do exercise 6 with 18! replaced by 26!

*8. Five books are to be arranged on a shelf.
   - a) How many permutations are possible?
   - b) How many permutations are possible if Romeo and Juliet must be next to The Catcher in the Rye? $(2 \cdot 4! = 48)$

*9. Challenge problem: Rubik's Cube was invented in the late 1970's by Erno Rubik to show his architectural students some properties in spatial relations.

1) Calculate how the advertisers can claim more than 3 billion permutations.
2) Calculate (explain) how a mathematician can claim more than 43 quintillion color patterns.

Lesson 53 - Permutations of n Things Taken r at a Time

AIM: How do we solve permutation problems involving n things taken r at a time?

PERFORMANCE OBJECTIVES: The students will be able to...

1. Employ the notation $_nP_r$.
2. Solve problems involving n things taken r at a time (calculator applications).
3. Practice problem solving techniques on permutation problems involving:
   a) win, place and show in a horse race.
   b) letter words from n given letters, election of r class officials from a class of n students.

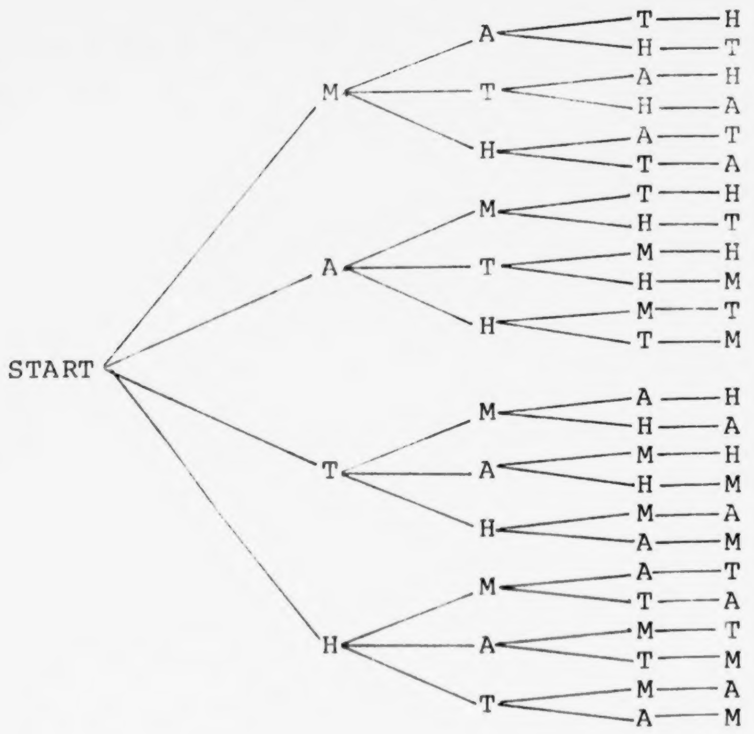
VOCABULARY: no new terms

MOTIVATION:

In how many ways may four books be arranged on a shelf? List all possible arrangements, using the letters M,A,T,H to represent the books.

DEVELOPMENT (Include pivotal questions):

1. Ask: How could we determine the number of such arrangements?" Demonstrate how a systematic approach (such as a tree diagram) is used to make certain we have them all.

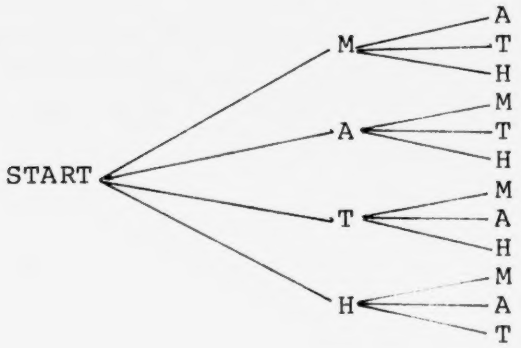


$4! = 24$

2. Review permutations as arrangements in which order counts. Refer to the fundamental counting principle to establish computation. then ask, "What symbol have we used that means the same as 4x3x2x1?" Thus, the number of permutation of n things when all are used is n! The number is also denoted by ${}_nP_n$. Apply to a number of problems leading to ${}_3P_3$, ${}_5P_5$, etc.

3. Now return to the motivation problem and ask, "In how many ways could two of these books be read?" (In other words, how many different permutations are possible if only two letters are used? i.e., MT, TM, AH, etc.) Elicit that there are fewer here (since only the first two stages of the tree must be considered), that is, $4 \times 3 = 12$. Introduce the symbol ${}_nP_r$ and compare to ${}_nP_n$. Thus, ${}_4P_4 = 4! = 4x3x2x1$, but ${}_4P_2 = 4 \times 3$ (the last two factors are omitted). See diagram. Note that there are twice as many branches in the tree corresponding to the use of four letters (${}_4P_4 = 4x3x2x1 = 4!$) as there are in the tree corresponding to the use of only two letters

$$({}_4P_2 = \frac{4\times3\times2\times1}{2\times1} = \frac{4!}{2!} = 4\times3).$$



4 x 3 = 12

4. Evaluate and compare ${}_5P_2$ and ${}_5P_5$. The former is $5 \cdot 4 = 20$; the latter is 5! The former is missing the last three factors (3 = 5 - 2). Generalize that ${}_nP_r$ is "missing" the last n-r factors, so

$${}_nP_r = \underbrace{(n)(n-1)(n-2)\ldots(n-r-1)}_{\longleftarrow r \text{ factors} \longrightarrow}.$$

Thus, ${}_nP_r = \frac{nPn}{(n-r)!} = \frac{n!}{(n-r)!}$

While it is important for students to be aware of this formula, they should be encouraged to use counting methods to solve permutation problems.

APPLICATIONS:

1. How many ways may three horses finish if there are only 3 in a race? (3!) If there are 10 in a race? (${}_{10}P_3 = 10\times9\times8$) Discuss "win, place and show".
2. Assign a problem on the election of 4 officers from a class of 30 students, etc.

   $${}_{30}P_4 = 30 \times 29 \times 28 \times 27 \text{ or } \frac{30!}{26!}$$
3. You and a friend agree to use colored flags to send secret messages to each other. If each of you has a similar set of seven flags of different colors, and each "message" uses three flags, how many different "messages" can either of you send?
4. In the situation described in problem 3, how many messages are possible if the blue flag must be in the middle?

SUMMARY:

1. What is a permutation?
2. What is meant by ${}_nP_n$, ${}_nP_r$? What are formulas for these?
3. A manufacturer produces "combination locks", each of which has 40 numerals on its dial. If each "combination" involves 3 numbers, how many locks can he make without duplicating the formula?

HOMEWORK:

1. Express in terms of factorials:

   a) ${}_7P_7$ b) ${}_7P_6$ c) ${}_7P_5$ d) ${}_7P_1$

2. Compute the following:

   a) ${}_3P_3$ b) ${}_4P_3$ c) ${}_5P_3$ d) ${}_6P_3$

3. How many different 2-digit numbers can be formed using the digits 5,6,7,8,9 (use no digit more than once)?
4. How many different 3-letter "words" can be formed from the letters C,O,M,P,U,T,E,R (use no letter more than once)?
5. How many different "words" may be formed by rearranging the letters of:

   a) MEAT c) MEATS

   *b) MEET *d) MEETS

Lesson 54 - Probability w/wo Replacement

AIM: How do we calculate probabilities involving replacement of objects?

PERFORMANCE OBJECTIVES: The students will be able to...

1. Apply problem solving techniques to problems involving replacement and non-replacement of objects.
2. Apply counting principle and tree diagrams to problems involving replacement of objects.

VOCABULARY: urn, replacement

MOTIVATION:

Five students (3 boys and 2 girls) have each been nominated to represent the school on two separate special occasions. In each case, the selection committee has decided to choose the representative of the school by placing all five names in a hat and drawing one out.

a) What is the probability that, for each of the special events, the school will be represented by a boy?

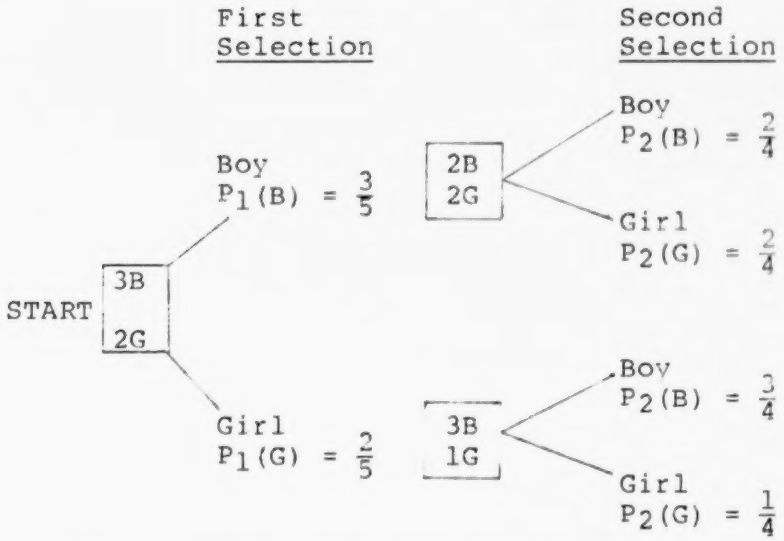
b) How will this answer change if the principal announces that the same student cannot represent the school at both events?

DEVELOPMENT (Include pivotal questions):

1. Discuss motivation. (The selection process can be represented by drawing marbles from an urn - 3 blue and 2 green.)

   In case a, it is possible for the same student to be chosen twice, since his name is "replaced" in the urn after being drawn (selection of the representatives for the two occasions are independent activities). This is a compound event, and students may use multiplication of

probabilities to get the answer. $(\frac{3}{5}\cdot\frac{3}{5} = \frac{9}{25})$
In case b, because of the principal's rule, the second representative will be chosen from a field of four candidates. This corresponds to an urn problem "without replacement". A tree diagram will be useful:


First Selection
Second Selection
START
3B
2G
Boy
$P_1(B) = \frac{3}{5}$
2B
2G
Boy
$P_2(B) = \frac{2}{4}$
Girl
$P_2(G) = \frac{2}{4}$
Girl
$P_1(G) = \frac{2}{5}$
3B
1G
Boy
$P_2(B) = \frac{3}{4}$
Girl
$P_2(G) = \frac{1}{4}$

Again, elicit that probabilities should be multiplied to get the answer $(\frac{3}{5}\cdot\frac{2}{4} = \frac{6}{20}$ or $\frac{3}{10})$.
Students should observe the difference between problems where replacement is permitted and not. When replacement is not permitted, the sample space is diminished.

2. Apply to license plate numbers where numbers or letters may be repeated. Ask: If a plate consists of 5 letters, what is the total number of plates that could be issued? (26^5) This is a permutation with repetitions. (If repititions were not allowed, the answer would be ${}_{26}P_5$.)
Ask: What is the probability of a random plate starting with MATH? Answer: since there are 26 such plates,

$$P(E) = \frac{26}{26^5} \text{ or } \frac{1}{26^4}$$

Ask: What is the probability of a plate which begins with the letter Q? Answer: $P(E) = \frac{26^4}{26^5} = \frac{1}{26}$

3. Bob keeps his socks loose in the top drawer of his bureau. In the dark, he selects two at random. If the

drawer contained 5 red socks and 4 black ones, what is the probability that he has drawn a matched pair?

$$P(\text{two reds}) = \frac{5}{9}\cdot\frac{4}{8} = \frac{20}{72}$$

$$P(\text{two blacks}) = \frac{4}{9}\cdot\frac{3}{8} = \frac{12}{72}$$

In either case (two reds or two blacks), we have a matched pair; therefore, we should add the probabilities. Answer = $\frac{32}{72}$ or $\frac{4}{9}$.

Ask: How many socks must Bob take from the drawer to make this probability equal 1 (certainty of a matched pair)? Answer: 3 socks.

APPLICATIONS:

Apply to problems involving spinners.

"Why are spinner problems of the 'replacement type'?"

SUMMARY:

If selection is made with or without replacement, how will the probability be affected?

In problems involving replacement of objects, the probability of each event does not change. With no replacement, the probability changes after each event occurs.

HOMEWORK:

1. Show the sample space tree, then compute the probability:
   A bag contains 5 slips of paper marked g, g, r, r and w.
   a) What is the probability of drawing a slip marked
      1) g 2) r 3) w
   b) What is the probability, with replacement, of drawing:
      1) g and g 3) g and w
      2) g and r 4) w and w
   c) What is the probability, without replacement, of drawing:
      1) g and g 3) g and w
      2) g and r 4) w and w

2. Each of two bags contains 5 marbles, 3 red and 2 white. a marble is randomly selected from the first bag and placed into the second bag. A marble is then drawn at random from the first bag. What is the probability that this marble will be: a) red $(\frac{6}{10})$ b) white? $(\frac{4}{10})$ Suppose instead, the marble were drawn from the second bag. What is the probability that the marble drawn is:
   c) red $(\frac{3}{5})$ d) white $(\frac{2}{5})$
   e) the marble originally in the first bag? $(\frac{1}{6})$

3. Compute:
   a) $8! - 7!$
   b) $\frac{5! + 6!}{4!}$
   c) ${}_7P_3 - {}_7P_2$
   d) ${}_8P_3 - {}_7P_3$

4. How many different words are possible by rearranging the letters in:
   a) HOME
   b) SWEET
   c) SOLO
   d) SOLOS

5. One bag has 6 slips marked 1,2,2,3,3,3. A second bag has 5 slips marked 4,4,5,5,6. A slip is drawn from each bag. What is the probability that:
   a) the numbers are both even.
   b) the numbers are both odd.
   c) one number is even and the other is odd.
   d) the sum of the numbers is 7.

Lesson 55 - Introduction to Statistics

AIM: What do we mean by statistics in mathematics?

PERFORMANCE OBJECTIVES: The students will be able to...

1. Define statistics as a discipline dealing with the collection, organization and interpretation of numerical information called data.
2. Examine sampling techniques and principles used by statisticians.

VOCABULARY: statistics, data, survey, sample, random, descriptive statistics, distribution, population, inferences

MOTIVATION:

A federal agency is called the Bureau of Labor Statistics. Ask, "What do you think that agency does?" Discuss statistics on unemployment, production, distribution of incomes, etc. Ask, "What other statistics are currently in the news?" (Sports, stock prices, etc.)

DEVELOPMENT:

1. These are examples of descriptive statistics - numbers obtained by collecting and organizing data. These are then used to draw conclusions or inferences about the rest of the "population". To collect data, a "sample" is selected, often obtained through a survey. Ask, "What is a survey?" Give examples of fair and unfair samples. Ask, "What do you think is meant by random sample?" Discuss a procedure of generating a list of Social Security numbers (obtained, for instance, by throwing dice or using a computer) in order to obtain "random" income information. Incomes of subscribers to the Wall Street Journal would not constitute a random sample

(because their incomes would likely be higher than the population in general).

2. Examples of sampling may be given. For instance, the height of a student in this class is a sample which may or may not be close to the average height of students in the school. A better sample would contain the heights of more students in more classes. Ask, "Why would the average of these numbers probably be closer to the average in the school?"

3. A major function of statistics is to interpret data and make predictions based on that data. Give examples of TV ratings, pre-election polls*, the use of SAT scores by colleges, etc. The relation of statistics to probability may be noted here.

   *Mention may be made of the Literary Digest poll in 1936 which predicted Landon over Roosevelt in the presidential election. The sample was far from random in that only subscribers or people who owned telephones (in a depression year) were polled.

4. Have students suggest ways of organizing data such as the heights of the students in the class. Inform them that this will be the topic of the next lesson.

MATERIALS: articles, charts from newspapers, magazines

APPLICATIONS:

1. How would you organize a survey to determine how many Canadians are bi-lingual?

2. What interests you about your classmates? Design a questionnaire which could be used to collect data about areas of interest which are suggested.

SUMMARY:

1. Ask students to explain, in their own words, the meaning of each of the terms included in the vocabulary list.

2. What statistics have you recently read about or seen on TV?

3. Why is the study of statistics important?

HOMEWORK:

1. Select a page in some book that has many lines of written material. Make a tally of the numbers of vowels - a,e,i,o,u - that appear on this page. Determine how many letters there are all told on this page. What fractional part of the total number of letters is each of these vowels: a,e,i,o,u?

2. Find your last name in the telephone directory. Make a tally of the second vowel of each last name on this page.

3. Bring in a newspaper or magazine item which includes statistics: what they are; how they were obtained; how they are interpreted.

4. What statistics could be compiled about:
   a) Presidents of the U.S.
   b) U.S. colleges
   c) automobiles

Lessons 56 - Statistical Data, Tables and Graphs

AIM: How do we organize data into tables?

PERFORMANCE OBJECTIVES: The students will be able to...

1. Use tallies and properly selected intervals to transform unorganized data (set of test scores) into a table of grouped data.
2. List rules for grouping data into tables as a preliminary to re-examining picture graphs, bar graphs and histograms.

VOCABULARY: tables, frequency, intervals, continuous, discrete, tallies

MOTIVATION:

You have volunteered to prepare a report about TV viewing habits in your community. Eighty families kept journals over an eight-week period. The list below shows the total number of viewing hours for each of the families for the time period of the survey. What can you conclude?

161	172	157	160	188	176	180	201
193	195	162	138	210	162	161	176
163	119	180	140	154	114	162	166
194	157	156	166	238	138	170	149
126	219	172	173	170	222	169	210
170	164	190	180	153	158	155	129
171	162	160	152	158	168	173	169
166	141	216	122	184	156	156	155
210	168	167	165	162	196	128	169
163	150	101	174	165	156	130	178

DEVELOPMENT (Include pivotal questions):

1. Discuss motivation. Students should observe that every family in this survey spent at least 100 hours watching TV (about $12\frac{1}{2}$ hours per week). Minimum and maximum numbers reported were 101 and 238.
   Pivotal question: What was the behavior of a typical

family?

Elicit: The answer is not obvious.

Pivotal question: How may we organize this data to make patterns clearer?

Students will suggest arranging the values in size order.

Ask: There are more than a hundred values in the range; what can be done to simplify the counting process?

Students should suggest tallying values within intervals. Discuss the appropriate size for intervals in this case; 25 is a useful compromise, showing neither too little nor too much detail. Intervals must be of equal length with no gaps or overlaps, and together include all possible values.

2. Have a student record tally marks on the board.

Interval	Tally	Number (frequency)
100-124	////	4
125-149	~~////~~ ////	9
150-174	~~////~~ ~~////~~ ~~////~~ ~~////~~ ~~////~~ ~~////~~ ~~////~~ ~~////~~ ~~////~~ /	46
175-199	~~////~~ ~~////~~ ///	13
200-224	~~////~~ //	7
225-249	/	1

Define "frequency" as the number of tallies in each interval.

Ask: Can we now comment about the behavior of a "typical family"?

Students should observe that most families in this community watched TV for 150-174 hours during the eight-week period of the survey.

3. Ask: How many families in this group watched 200 or more hours? (8) Fewer than 150 hours? (13)

Ask: If you saw only the frequency table, could you tell exactly how many families reported exactly 160 hours of viewing? The minimum or maximum values reported?

The class should conclude that a frequency table makes distribution patterns obvious at a glance, but conceals some specific information.

4. Ask the students to make a frequency table for the following data:

Heights of Students in a 9th Grade Math Class

61	64	62	63	64	59
71	69	58	67	66	62
64	62	64	62	61	65
60	62	71	59	64	62
65	67	60	61	72	73
71	63	65	68	62	67

The class should discuss the appropriate size for an interval in this case; it is recommended that two (58-59, 60-61, etc.) or three (58-60, 61-63, etc.) be used. The teacher may wish to divide the class into teams with different teams using different intervals.

Ask the students to draw conclusions about the data.

APPLICATIONS:

1. Make a frequency table, using the most recent test scores from the class.
2. Make a frequency table for TV viewing habits for this class. Ask: How many hours does your family watch TV each week?

SUMMARY:

1. What is the value of a frequency table?
2. What guidelines must be used in specifying appropriate intervals for grouping data?
3. Do you think the community presented in the motivation problem is similar to the one we live in?

HOMEWORK:

1. Provide data for one of the following and ask students to make a frequency table and draw conclusions:
   a) last year's Course One Regents test results for the school
   b) heights of players on the basketball team
   c) weights of players on the football team

2. Ask students to conduct a survey, express the results in a frequency table and draw conclusions about one of the following:
   a) number of hours spent on homework each night
   b) number of movies seen in a two-month period
   c) number of people in family

Lessons 57 - Statistical Data, Tables and Graphs

AIM: How do we organize data into graphs and histograms?

PERFORMANCE OBJECTIVES: The students will be able to...

1. Present, analyze and interpret data in picture graphs, bar graphs and histograms.
2. Construct graphs to show organized data.

VOCABULARY: picture graph, bar graph, circle graph, histogram

MOTIVATION:

Show the class a pictograph (picture graph), bar graph or histogram from a recent newspaper or magazine. Ask, "Why is this better than just showing the data?"

DEVELOPMENT (Include pivotal questions):

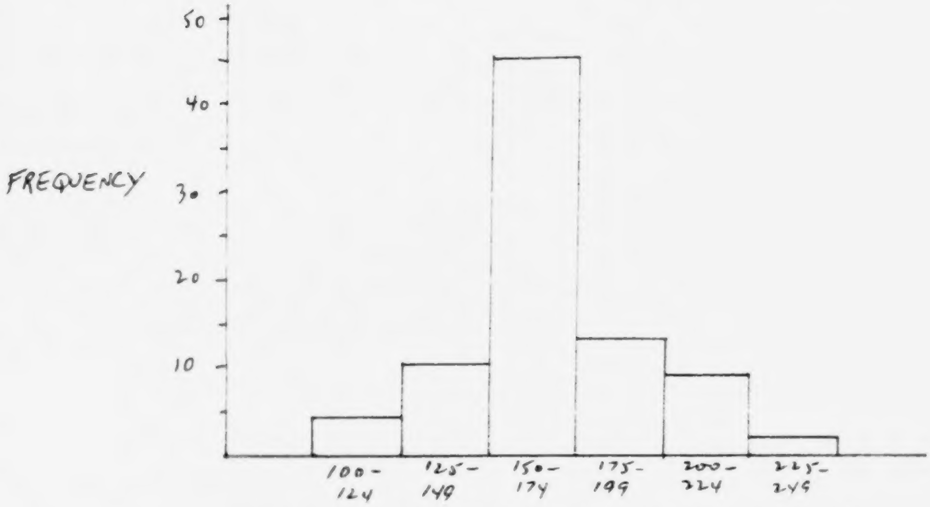
1. Discuss motivation. Students should observe that graphs are more interesting and draw attention visually to the most dramatic statistical figures.
   Ask: What other kinds of graphs have you seen used in this way?
   Elicit: pictograph, bar graph, line graph, circle graph, histogram. The teacher should present samples of each type.
2. Briefly, mention the strengths of each type:
   a) In a pictograph, an appealing symbol is used to represent a given quantity of something. It is obvious at a glance what topic the graph discusses.
   b) A circle graph dramatically shows the distribution of values among parts of a whole. (These are frequently used to illustrate budgets.)
   c) A line graph is especially useful to reveal changes (or trends) over time. (Hospital temperature chart, company sales performance are good

illustrations.)

d) A bar graph facilitates comparison of any two values displayed.

e) A histogram is a vertical bar graph which illustrates frequencies for intervals of continuous data. It has no spaces between the bars; the bars are placed together to emphasize the idea of continuity: one interval begins where the previous one ends.

3. The class should now construct a histogram based on the data from the motivation problem of Lesson 56.



NUMBER OF TV-VIEWING HOURS PER FAMILY

4. Ask students to respond to the questions such as these, basing their answers solely on information which can be drawn from the histogram:

a) What was the behavior of a typical family?

b) What drew the fewest responses?

c) How many families reported watching fewer than 200 hours?

d) What percent of the families surveyed lie in the interval that drew the most responses?

APPLICATIONS:

1. Construct a histogram based on the student heights problem of Lesson 56.
2. Construct histograms based on the applications problems of Lesson 56.

SUMMARY:

1. Why are the bars placed against each other in a histogram?
2. What are the special advantages of a histogram (compared to other types of graphs)?
3. Identify another situation in which you might wish to use a histogram to communicate information.
4. Suppose you collected data about what type of pet each student in the school owned. Would you use a histogram to display this information? Why or why not?

HOMEWORK:

1. Construct histograms based on the projects described in the homework in Lesson 56.
2. Bring in a histogram taken from a newspaper or magazine. Write several conclusions about the data presented.
3. Bring in a pictograph, line graph, circle graph or bar graph taken from a newspaper or magazine. Write several conclusions about the data presented. Why was this form of graph chosen, rather than a histogram?

Lessons 58 - Mean, Median, Mode

AIM: What do we mean by measures of central tendency?

PERFORMANCE OBJECTIVES: The students will be able to...

1. Calculate and interpret (from organized and unorganized data) mean, median and mode (Calculator Applications).
2. Solve verbal problems involving mean, median and mode.

VOCABULARY: mean, median, mode, inference, central tendency, bimodal

MOTIVATION:

Helen's test scores so far this term have been 82, 78, 83, 81, 75, 90, 83. What single grade would you use to represent her test performance in this class?

DEVELOPMENT (Include pivotal questions):

1. Discuss motivation. Students will probably be familiar with the process of "finding the average" by adding all scores and dividing by the number of tests. Identify the value found in this way ($81\frac{5}{7}$) as the mean of this set. Ask: What score did she receive most frequently? (83 is the mode.)
   Ask: Is either of these numbers in the middle of the group? (No, if the scores were placed in size order, 82 would be in the middle; this value is called the median.) Each of these numbers expresses a different kind of central value for the set. Teachers usually use the mean, which represents the grade Helen would have received on every test, if her performance (and the difficulty of the tests!) was uniform.
2. Five people have the following annual salaries: \$20,000, \$22,000, \$25,000, \$38,000 and \$63,000. Find the mean, mode and median salary.

Is the mean ($33,600) really a good number to use as typical in representing these five salaries? Discuss the shortcoming of using the mean when there are very large (or small) numbers which are quite different from the other numbers.

The median ($25,000) is a better expression of a typical annual salary for this group. This set has no mode.

3. Ask, "What number could represent the median for an even number of terms such as 8,6,3,7,7,5?" (The median is $\frac{6+7}{2} = 6.5$).

4. Discuss the advantage of knowing the number which occurs most frequently (the mode). This number could be used by a clothing store in ordering, for example, sizes.
   Give an example of a set which is bimodal:
   42,87,63,28,87,19,43,87,42,42

APPLICATIONS:

1. If the sum of several items is 816 and the mean is 48, how many terms are in the data?
2. In the motivation problem, what mark does Helen need on the eighth test to bring her average (mean) up to 83?
3. For which set of numbers are the mean, mode and median all equal?
   $\{2,2,5\}$, $\{2,5,5\}$, $\{2,2,5,5\}$, $\{2,3,4,4,7\}$
4. How would the mean be affected if each term in a distribution was doubled? ...was increased by 3?

SUMMARY:

1. Name and explain three ways to express the center of a group of data.
2. Give examples of situations where you would be most interested in the mode.
3. Repeat question 2 for median and mean.

4. The average (mean) height of a group of seven students is 63". Ann, who is 65" tall, joins the group. What is the new mean average height?

HOMEWORK:

1. For each set of data used in Lessons 56 and 57:
   a) Calculate the mean, mode and median.
   b) Which of these values best expresses a typical value of the set?

2. Calculate the mean, mode and median:
   a) 19,20,17,24,29
   b) 47,85,300,85,53,47
   c) 42,43,44,43,40,49
   d) -7,-3,2,-1,-3,0
   e) 18.7,19.4,17.25,16.8,18.3,19.2,17.1,19.4
   f) $2\frac{1}{2}, 4\frac{1}{4}, 9\frac{3}{4}, 5, 2\frac{1}{2}$

3. Find the value(s) of x so that the set $\{x,5,4,7,4\}$ will have:
   a) 9 as its mean
   b) 0 as its mean
   c) two modes
   d) 5 as its median

Lessons 59 - Grouped Data

AIM: How do we use grouped data to determine measures of central tendency?

PERFORMANCE OBJECTIVES: The students will be able to...

1. Construct frequency tables with appropriate intervals.
2. Determine the interval containing the median and the modal interval from frequency distribution tables.

VOCABULARY: grouped data, frequency table, modal, interval

MOTIVATION:

These are the ages of people living in a particular apartment building:

36,22,3,41,32,10,68,5,11,39,22,35,12,5,9,57,37,59,42,33

1. Calculate the values of the mean, mode and median.
2. Make a frequency table with ten tallies in each interval (i.e., first interval is 0-9, next is 10-19...)

DEVELOPMENT (Include pivotal questions):

1. Discuss motivation problem. the mean is 28.9; the median is 32.5; the modes are 5 and 22 (the set is bimodal).

Age	Frequency
0-9	4
10-19	3
20-29	2
30-39	6
40-49	2
50-59	2
60-69	1

2. Ask: If you had been given the frequency table rather than the data, could you have calculated the values of mean, median and mode? (No)
   Ask: Could you have predicted which intervals would contain the modes? (No)

Define: The modal interval is the interval with the greatest frequency of data.

Ask: Which is the modal interval for this data? (30-39)

Note: In this case, the modal interval does not contain either mode.

Ask: Given only the frequency table, could you have predicted which interval would contain the mean or the median?

Mean: no. Consider the effect of replacing the values 32,41,42 of the original set with 39,49,49; the frequency table would be unchanged but the mean would move to a different interval.

Median: yes! The interval can be predicted using the following strategy: since there are twenty numbers in the set, the median must be the average of the 10th and 11th data elements (in size-order arrangement); since both of these lie in the 30-39 interval, the median will lie in this interval also. This strategy will always work when the number of elements is odd.

APPLICATIONS:

Using frequency tables made for problems considered previously (Lessons 56-58), ask students to identify the modal interval and the interval containing the median.

SUMMARY:

1. Will the mode lie in the modal interval? Explain.
2. Is it always possible to identify the interval containing the median from a frequency table? Explain.

HOMEWORK:

1. If data are grouped in an odd number of intervals, will the median always lie in the middle interval? Explain.
2. Give an example of grouped data for which the mode does not lie in the modal interval.

3. Give an example of grouped data for which the mode does *not* lie in the modal interval.

4. Make a new frequency table for the data in today's motivation problem, with *fifteen* tallies in each interval (0-14, 15-29, 30-44...). Which interval contains the mean? ...the median? ...the mode? Which is the modal interval? Does the modal interval contain the mean?

Lesson 60 - Cumulative Frequency Histograms

AIM: How do we construct and interpret cumulative frequency histograms?

PERFORMANCE OBJECTIVES: The students will be able to...

1. Construct cumulative frequency histograms from given sets of data.
2. Interpret cumulative frequency histograms to provide quartile and percentile scores.

VOCABULARY: cumulative frequency, quartiles, percentiles

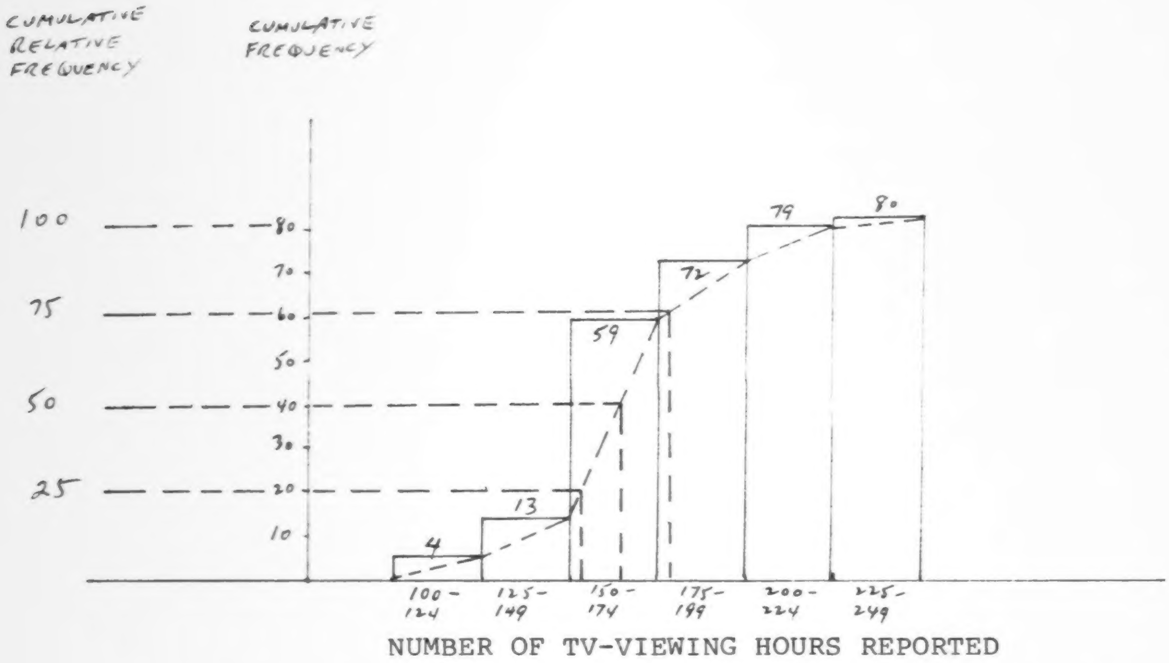
MOTIVATION:

Ask the students to refer to the data about TV viewing habits used in Lessons 56 and 57. What percentage of the families watched TV for 124 hours or fewer? What percentage of the families reported 149 or fewer hours?

DEVELOPMENT:

1. Discuss motivation problem. Four out of the 80 families reported 124 hours or less; this is 5%. To answer the second question, it is necessary to add the number of families in the 100-124 interval to the number in the 125-149 interval before dividing by 80; 17 out of 80 is $21\frac{1}{4}$%.
2. The answers to these (and similar) questions can be represented pictorially in a cummulative frequency histogram.
3. Define the total number of terms less than or equal to a given term as the cumulative frequency of that term, and the ratio of that cumulative frequency to the total number of terms as the cumulative relative frequency of the term.

4. Ask for the total number of families reporting less than or equal to 124 (149,174,199,224,249) hours of TV viewing, and construct a cummulative frequency histogram.



Elicit values for "cummulative relative frequency" by asking, "How many families would be 100% (50%, 25%) of the group involved in the survey?"

5. Certain statistically significant measures can be 'read off' the graph given above (see dotted lines beginning at the left of graph):

   a) The median (measure at or below which 50% of all the measures fall) weight of our group appears to be 165.

   b) The upper quartile (the measure at or below which 75% fall) appears to be 176.

   c) The lower quartile (the measure at or below which 25% fall) appears to be 151.

These approximations are based on the intersections of horizontal lines corresponding to cummulative relative frequencies of 50%, 75% and 25% with a dotted line graph joining successive upper right hand corners of the bars (a pictorial representation of the assumption that distribution within each interval is uniform).

6. It is also possible to determine percentile values. (The nth percentile is the measure at or below which n% of the measures fall.) For example, in our sample, the 13th percentile appears to be about 140. People often confuse the meaning of percentile and quartile, believing these terms to indicate ranges rather than specific measures, and it is suggested that attention be given to this problem.
7. Could the cumulative frequency histogram ever turn downward? Why? What is the percentile for the highest term?

APPLICATIONS:

Have the class construct cumulative frequency histograms from other sets of data and locate various quartiles and percentiles from the graph.

SUMMARY:

1. What does a cumulative frequency histogram show? A cumulative frequency histogram shows how the numbers in a set of data increase.
2. What does a percentile of 75 on the SAT mean?

HOMEWORK:

On a certain test, the distribution of scores ranged between 50 and 99. The grouped data table is shown below.

Interval	Number of Scores
50-59	2
60-69	10
70-79	47
80-89	23
90-99	18

Prepare:

a) a frequency histogram of the grouped data.

b) a cumulative frequency histogram of the grouped data.

c) Determine into which quartile the following scores fall.

1) 65 2) 75 3) 85 4) 95